Intersection Synchronous Logic

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Outline

- Motivation

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- Intuitionistic Logic

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- **IT**: type system.
- That is, it concerns terms and types.
- Goal: to give a proof-theoretical justification for **IT**.
- That is, reformulate Intersection Types **IT** by means of a pure logical system.
- Basis step: to clarify the difference between intersection $\cap$ and intuitionistic conjunction $\wedge$ by imposing constraints on the use of logical and structural rules of intuitionistic logic.

Motivation

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- Intersection can be introduced only between formulas typing the *same* term.
- **ISL**: never relies on λ-terms to mark the points where intersection operators of **IT** can be introduced.

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- Proves properties *of sets of deductions of* **NJ**.
- The deductions of the same set must be *synchronous with respect to the use of* $\longrightarrow$*-introduction and elimination*.
- Deductions $\Pi_1, \ldots, \Pi_n$ of **NJ** in the same set all have the same structure.
- Rules of **ISL** must inductively build the sets of synchronous derivations of **NJ** as they were *equivalence classes*.

Logic and Programming

The Curry-Howard isomorphism

Logical formulas	$\sim$	Types
Proofs	$\sim$	Programs
Cut elimination	$\sim$	Evaluation

Logic and Programming

From Logic to Programming

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- rigorous foundation for the design of Programming Languages
- tools for automatic synthesis, verification, transformation of programs

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- new problems and results in proof theory (e.g., typability problem)
- design of new logical systems inspired from programming (e.g., light logics)

Proofs decoration

$$(R)\frac{\phi_1, ..., \phi_n \vdash \psi_i \qquad (1 \leq i \leq n)}{\phi_1, ..., \phi_n \vdash R(\psi_1, ..., \psi_n)}$$

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$$\frac{x_1 : \phi_1, \ldots, x_n : \phi_n \vdash M_i : \psi_i \quad (1 \leq i \leq n)}{x_1 : \phi_1, \ldots, x_n : \phi_n \vdash R(M_1, \ldots, M_n) : R(\psi_1, \ldots, \psi_n)}$$

An unusual case study

- From Computations to proofs
  Given a type assignment, assigning types to terms, we asked for a logical foundation of it, i.e., for a logic such that the type assignment can be seen as a decoration of it.

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  Given a type assignment, assigning types to terms, we asked for a logical foundation of it, i.e., for a logic such that the type assignment can be seen as a decoration of it.
- and back
  By decorating such a logic with a different technique, we built a new typed language, for expressing the discrete polymorphism, which was a longstanding open problem.

$\{\wedge \rightarrow\}$-fragment of NJ

- *Formulae*:

$$\sigma, \rho, \tau ::= a \mid (\sigma \rightarrow \sigma) \mid (\sigma \wedge \sigma)$$

where a belongs to a denumerable set of constants.

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- A *context* is a *finite sequence* $\sigma_1, \ldots, \sigma_m$ of formulae. Contexts are denoted by Γ and Δ.

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- A *context* is a *finite sequence* $\sigma_1, \ldots, \sigma_m$ of formulae. Contexts are denoted by Γ and Δ.
- The implicative and conjunctive fragment of **NJ** proves statements $\Gamma \vdash_{\mathrm{NJ}} \sigma$, where Γ is a context and σ a formula.

$\{\wedge \rightarrow\}$-fragment of NJ

$$(A)\frac{}{\sigma \vdash_{\mathrm{NJ}} \sigma} \qquad (W)\frac{\Gamma \vdash_{\mathrm{NJ}} \sigma}{\Gamma, \tau \vdash_{\mathrm{NJ}} \sigma}$$

$$(X)\frac{\Gamma_1, \sigma_1, \sigma_2, \Gamma_2 \vdash_{\mathrm{NJ}} \sigma}{\Gamma_1, \sigma_2, \sigma_1, \Gamma_2 \vdash_{\mathrm{NJ}} \sigma} \qquad (\wedge I)\frac{\Gamma \vdash_{\mathrm{NJ}} \sigma \quad \Gamma \vdash_{\mathrm{NJ}} \tau}{\Gamma \vdash_{\mathrm{NJ}} \sigma \wedge \tau}$$

$$(\wedge E^l)\frac{\Gamma \vdash_{\mathrm{NJ}} \sigma \wedge \tau}{\Gamma \vdash_{\mathrm{NJ}} \sigma} \qquad (\wedge E^r)\frac{\Gamma \vdash_{\mathrm{NJ}} \sigma \wedge \tau}{\Gamma \vdash_{\mathrm{NJ}} \tau}$$

$$(\rightarrow I)\frac{\Gamma, \sigma \vdash_{\mathrm{NJ}} \tau}{\Gamma \vdash_{\mathrm{NJ}} \sigma \rightarrow \tau} \qquad (\rightarrow E)\frac{\Gamma \vdash_{\mathrm{NJ}} \sigma \rightarrow \tau \quad \Gamma \vdash_{\mathrm{NJ}} \sigma}{\Gamma \vdash_{\mathrm{NJ}} \tau}$$

Decorating NJ

$$(A)\frac{}{x : \sigma \vdash^{*}_{\mathrm{NJ}} x : \sigma}$$

$$(W)\frac{\Gamma \vdash^{*}_{\mathrm{NJ}} M : \sigma \quad x \notin dom(\Gamma)}{\Gamma, x : \tau \vdash^{*}_{\mathrm{NJ}} M : \sigma}$$

$$(X)\frac{\Gamma_1, x : \sigma_1, y : \sigma_2, \Gamma_2 \vdash^{*}_{\mathrm{NJ}} M : \sigma}{\Gamma_1, y : \sigma_2, x : \sigma_1, \Gamma_2 \vdash^{*}_{\mathrm{NJ}} M : \sigma}$$

$$(\wedge I)\frac{\Gamma \vdash^{*}_{\mathrm{NJ}} M : \sigma \quad \Gamma \vdash^{*}_{\mathrm{NJ}} N : \tau}{\Gamma \vdash^{*}_{\mathrm{NJ}} (M, N) : \sigma \wedge \tau}$$

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- The two formulations are equivalent.
- Since we are interested to explore the structures of the proofs, we need to express explicitly the structural rules.

Properties of IT

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- **IT** is undecidable
- **IT** has the principal typing property:
  if a term M can be typed then it has a principal typing such that all and only their typings can be obtained from it by means of suitable operations

The problem

Is there a logical foundation for IT?

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i.e.

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is there a logic such that IT can be obtained from it through a decoration?

Refining NJ

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- $\&$: *asynchronous conjunction* that gives type to the pair (M, N), since M and N are distinct.
- Synchronous conjunction and the intersection have the same symbol: the two connectives are strongly related.

NJr

$$(A)\,\frac{}{x:\sigma \vdash_{\mathrm{NJr}} x:\sigma}$$

$$(W)\,\frac{\Gamma \vdash_{\mathrm{NJr}} M:\sigma \quad x \notin dom(\Gamma)}{\Gamma, x:\tau \vdash_{\mathrm{NJr}} M:\sigma} \qquad (X)\,\frac{\Gamma_1, x:\sigma_1, y:\sigma_2, \Gamma_2 \vdash_{\mathrm{NJr}} M:\sigma}{\Gamma_1, y:\sigma_2, x:\sigma_1, \Gamma_2 \vdash_{\mathrm{NJr}} M:\sigma}$$

$$(\cap I)\,\frac{\Gamma \vdash_{\mathrm{NJr}} M:\sigma \quad \Gamma \vdash_{\mathrm{NJr}} M:\tau}{\Gamma \vdash_{\mathrm{NJr}} M:\sigma \cap \tau} \qquad (\& I)\,\frac{\Gamma \vdash_{\mathrm{NJr}} M:\sigma \quad \Gamma \vdash_{\mathrm{NJr}} N:\tau}{\Gamma \vdash_{\mathrm{NJr}} (M,N):\sigma \& \tau}$$

$$(\cap E^l)\,\frac{\Gamma \vdash_{\mathrm{NJr}} M:\sigma \cap \tau}{\Gamma \vdash_{\mathrm{NJr}} M:\sigma} \qquad (\cap E^r)\,\frac{\Gamma \vdash_{\mathrm{NJr}} M:\sigma \cap \tau}{\Gamma \vdash_{\mathrm{NJr}} M:\tau}$$

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- $\cap$ merges sub-deductions where $\rightarrow$ is introduced or eliminated in the “same points”, namely, up to the use of the two kinds of conjunctions.
- **IT** is a sub-system of **NJr** where only synchronous conjunction is used.
- **ISL** gets rid of λ-terms to get the same properties as **IT**.

The logical system ISL

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- An *atom* is a pair $\mathcal{A} : (\Gamma; \alpha)$.
- *Molecule* $\mathcal{M} = [\mathcal{A}_1, \ldots, \mathcal{A}_n]$: a finite multiset of atoms such that the contexts in all atoms have the same cardinality.

ISL

$$\frac{}{[(\alpha_i;\alpha_i) \mid 1 \leq i \leq r]}\ (A) \qquad \frac{\mathcal{M} \cup \mathcal{N}}{\mathcal{M}}\ (P)$$

$$\frac{[(\Gamma_i;\beta_i) \mid 1 \leq i \leq r]}{[(\Gamma_i,\alpha_i;\beta_i) \mid 1 \leq i \leq r]}\ (W) \qquad \frac{[(\Gamma_1^i,\beta_i,\alpha_i,\Gamma_2^i;\sigma_i) \mid 1 \leq i \leq r]}{[(\Gamma_1^i,\alpha_i,\beta_i,\Gamma_2^i;\sigma_i) \mid 1 \leq i \leq r]}\ (X)$$

$$\frac{[(\Gamma_i,\alpha_i;\beta_i) \mid 1 \leq i \leq r]}{[(\Gamma_i;\alpha_i \to \beta_i) \mid 1 \leq i \leq r]}\ (\to I)$$

$$\frac{[(\Gamma_i;\alpha_i \to \beta_i) \mid 1 \leq i \leq r] \quad [(\Gamma_i;\alpha_i) \mid 1 \leq i \leq r]}{[(\Gamma_i;\beta_i) \mid 1 \leq i \leq r]}\ (\to E)$$

$$\frac{[(\Gamma_i;\alpha_i) \mid 1 \leq i \leq r] \quad [(\Gamma_i;\beta_i) \mid 1 \leq i \leq r]}{[(\Gamma_i;\alpha_i \& \beta_i) \mid 1 \leq i \leq r]}\ (\& I)$$

ISL

$$\frac{[(\Gamma_i;\alpha_i\&\beta_i)\mid 1\leq i\leq r]}{[(\Gamma_i;\alpha_i)\mid 1\leq i\leq r]}\ (\&E_L) \qquad \frac{[(\Gamma_i;\alpha_i\&\beta_i)\mid 1\leq i\leq r]}{[(\Gamma_i;\beta_i)\mid 1\leq i\leq r]}\ (\&E_R)$$

$$\frac{\mathcal{M}\cup[(\Gamma;\alpha),(\Gamma;\beta)]}{\mathcal{M}\cup[(\Gamma;\alpha\cap\beta)]}\ (\cap I)$$

$$\frac{\mathcal{M}\cup[(\Gamma;\alpha\cap\beta)]}{\mathcal{M}\cup[(\Gamma;\alpha)]}\ (\cap E_L) \qquad \frac{\mathcal{M}\cup[(\Gamma;\alpha\cap\beta)]}{\mathcal{M}\cup[(\Gamma;\beta)]}\ (\cap E_R)$$

Example

$$[(\alpha, \beta; \alpha), (\alpha, \beta; \beta)]$$

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$$[(\alpha,\beta;\alpha),(\alpha,\beta;\beta)]$$

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$$\begin{array}{cl} [(\alpha;\alpha)],[(\beta;\beta)] & \\ \hline\hline [(\alpha,\beta;\alpha)],[(\beta,\alpha;\beta)] & (W) \\ \hline [(\alpha,\beta;\alpha)],[(\alpha,\beta;\beta)] & (X) \\ \hline [(\alpha,\beta;\alpha\&\beta)] & (\&I) \end{array}$$

...and back

By decorating ISL, we obtain a typed programming language for discrete polymorphism, a longstanding open problem.

ISL and NJ

- Let $\mathcal{M}_i = [(\Gamma^i_1; \alpha^i_1), \ldots, (\Gamma^i_{m_i}; \alpha^i_{m_i})]$ for $1 \leq i \leq n$. Then

$$\vdash_{\textbf{ISL}} M_1 : (\mathcal{M}_1)^* \ldots M_n : (\mathcal{M}_n)^*$$

if and only if

$$\Gamma^i_j \vdash_{\mathrm{NJr}} M_i : \alpha^i_j$$

That is, a molecule represents a set of synchronous proofs of **NJ**.

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  That is, a molecule represents a set of synchronous proofs of **NJ**.
- **ISL** is a logic internalizing the difference between synchronicity and asynchronicity in **NJ**.

ISL and IT

- Let $\mathcal{M}_i = [(\Gamma^i_1; \alpha^i_1), \ldots, (\Gamma^i_{m_i}; \alpha^i_{m_i})]$ for $1 \leq i \leq n$ and suppose that $\vdash_{\textbf{ISL}} M_1 : (\mathcal{M}_1)^* \ldots M_n : (\mathcal{M}_n)^*$ where M_i doesn't have any occurrence of π_1, π_2 or $(.,.)$ and $\mathcal{M}_i$ doesn't have any occurrence of the connective $\wedge$. Then

$$\Gamma^i_j \vdash_{\textbf{IT}} M_i : \alpha^i_j$$

ISL and IT

- Let $\mathcal{M}_i = [(\Gamma_1^i; \alpha_1^i), \ldots, (\Gamma_{m_i}^i; \alpha_{m_i}^i)]$ for $1 \leq i \leq n$ and suppose that $\vdash_{\textbf{ISL}} M_1 : (\mathcal{M}_1)^* \ldots M_n : (\mathcal{M}_n)^*$ where M_i doesn't have any occurrence of π_1, π_2 or $(.,.)$ and $\mathcal{M}_i$ doesn't have any occurrence of the connective $\wedge$. Then

$$\Gamma_j^i \vdash_{\textbf{IT}} M_i : \alpha_j^i$$

- $\Gamma \vdash_{\textbf{IT}} M : \alpha$ implies $\vdash_{\textbf{ISL}}^* M : [(\Gamma)^*; \alpha]$

ISL and IT

- Let $\mathcal{M}_i = [(\Gamma^i_1; \alpha^i_1), \ldots, (\Gamma^i_{m_i}; \alpha^i_{m_i})]$ for $1 \leq i \leq n$ and suppose that $\vdash_{\textbf{ISL}} M_1 : (\mathcal{M}_1)^* \ldots M_n : (\mathcal{M}_n)^*$ where M_i doesn't have any occurrence of π_1, π_2 or $(.,.)$ and $\mathcal{M}_i$ doesn't have any occurrence of the connective $\wedge$. Then

$$\Gamma^i_j \vdash_{\textbf{IT}} M_i : \alpha^i_j$$

- $\Gamma \vdash_{\textbf{IT}} M : \alpha$ implies $\vdash^*_{\textbf{ISL}} M : [(\Gamma)^*; \alpha]$
- **ISL** is strongly normalizable.

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- Hence the system **ISL** gives a nice way of describing conjunction: it is a connective that has an “asynchronous” behavior.

Future work

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ISL	λ-calculus
IT	Strongly Normalizing λ-terms