Part I

Logics for Approximate Reasoning

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What this talk is NOT about...

- Fuzzy Logics
- Probabilistic Logics
- Multivalued Logics
- Intuitionistic, Relevant, Linear Logics.

General Contents

Contents

1. Motivation

- Approximations deal with hard problems.
  - Classical Propositional Satisfiability and Theorem Proving are hard (NP- and coNP-complete)
- Idealised agents are logically omniscient.
  - Real agents are limited.
  - Each step in an approximation models a limited agent.
- Approximations implicitly define notions of relevance.

1.1 History of Logics for Approximations

- Schaerf & Cadoli [1995]: Families of Logics S1 and S3, clausal form.
- Uses of S1, S3: diagnosis, belief revision.
- Finger & Wassermann [2001,2002,2005,2006]: Logics S3, s_1 for full propositional logics. Many kinds of approximation: The Universe of Approximations.
- Other approaches for approximation: Horn Clause Approximations
  - Linear approximation for an exponential problem.
- Our work follows the paradigm of Schaerf & Cadoli.

2. Intuitions of Approximations

- A family of Logics: $\mathsf{L}_1, \mathsf{L}_2, \ldots, \mathsf{L}_n$
- A target Logic L to approximate.
- The mathematical intuition:

$$\text{“} \lim_{n \to \infty} |\mathsf{L} - \mathsf{L}_n| = \varnothing \text{”}$$

(I know this expression has *no* formal meaning!)

2.1 Clarifying the Intuitions

- Think of L as $Th(\mathsf{L})$ or $\models_{\mathsf{L}}$.
- $|\mathsf{L} - \mathsf{L}_n| = (\mathsf{L} - \mathsf{L}_n) \cup (\mathsf{L}_n - \mathsf{L})$.
- The notion of approximation can be expressed as:

$$|\mathsf{L} - \mathsf{L}_1| \supset |\mathsf{L} - \mathsf{L}_2| \supset \cdots |\mathsf{L} - \mathsf{L}_n| \supset \cdots \supset \varnothing$$

- Theorem Proving: approximations "from below", $\mathsf{L}_n \subseteq \mathsf{L}$

$$\mathsf{L}_1 \subset \mathsf{L}_2 \subset \cdots \subset \mathsf{L}_n \subset \cdots \subseteq \mathsf{L}$$

- Theorem DisProving, SAT: approximations "from above", $\mathsf{L}_n \supseteq \mathsf{L}$

$$\mathsf{L}_1 \supset \mathsf{L}_2 \supset \cdots \supset \mathsf{L}_n \supset \cdots \supseteq \mathsf{L}$$

3. Schaerf & Cadoli's Proposal

- Restricted to Clausal Form: $\bigwedge(l_1 \vee \cdots \vee l_m)$. (Later in NNF)
- Based on a context set S.
- If $p \in S$, p behaves classically

$$v(p) = 1 \quad \text{iff} \quad v(\neg p) = 0$$

- If $p \notin S$, p has a special behaviour:

$$\left.\begin{array}{lll} v(p) = 0 & \text{and} & v(\neg p) = 1 \\ v(p) = 1 & \text{and} & v(\neg p) = 0 \\ v(p) = 1 & \text{and} & v(\neg p) = 1 \end{array}\right\} S_3(S)$$

$$\left.\begin{array}{lll} v(p) = 0 & \text{and} & v(\neg p) = 0 \end{array}\right\} S_1(S)$$

3.1 Approximate Entailment

- Logics S_3 are useful to approximate Theorem Proving:

$$B \models_S^3 \alpha \Longrightarrow B \models \alpha$$

- Logics S_1 are useful to approximate "Theorem Disproving" or SAT:

$$B \not\models_S^1 \alpha \Longrightarrow B \not\models \alpha$$

- When $S = \mathcal{P}$, $S_1(S) = S_3(S) = CL$.

- **Theorem 1** *There exists algorithms for deciding if* $B \models_S^3 \alpha$ *and deciding* $B \models_S^1 \alpha$ *which runs in* $O(|B|.|\alpha|.2^{|S|})$ *time.*

  For a fixed S these algorithms are linear!

4. Theorem proving in S_3

Example (due to [SC 95]).

Check whether $B \models \alpha$, where $\alpha = \neg\texttt{cow} \lor \texttt{molar-teeth}$ and

$$
\begin{aligned}
B = \quad & \{\neg\texttt{cow} \lor \texttt{grass-eater}, \neg\texttt{dog} \lor \texttt{carnivore}, \\
& \neg\texttt{grass-eater} \lor \neg\texttt{canine-teeth}, \neg\texttt{carnivore} \lor \texttt{mammal}, \\
& \neg\texttt{mammal} \lor \texttt{canine-teeth} \lor \texttt{molar-teeth}, \\
& \neg\texttt{grass-eater} \lor \texttt{mammal}, \neg\texttt{mammal} \lor \texttt{vertebrate}, \\
& \neg\texttt{vertebrate} \lor \texttt{animal}\}.
\end{aligned}
$$

For $S = \{\texttt{grass-eater}, \texttt{mammal}, \texttt{canine-teeth}\}$

4.1 S_3 simplification

- To decide whether $B \models^3_S \alpha$:
  - Delete from B all clauses which contain an atom $p \notin S$ that does not occur in α.
  - Obtain $B' \subseteq B$.
  - Apply classical theorem proving to the resulting $B' \models \alpha$.
  - $B' \models \alpha$ iff $B \models^3_S \alpha$.

4.2 S_3 Example (cont.)

Check whether $B \models \alpha$, where $\alpha = \neg\texttt{cow} \lor \texttt{molar-teeth}$ and

$$\begin{aligned} B = \quad & \{\neg\texttt{cow} \lor \texttt{grass-eater}, \neg\texttt{dog} \lor \texttt{carnivore}, \\ & \neg\texttt{grass-eater} \lor \neg\texttt{canine-teeth}, \neg\texttt{carnivore} \lor \texttt{mammal}, \\ & \neg\texttt{mammal} \lor \texttt{canine-teeth} \lor \texttt{molar-teeth}, \\ & \neg\texttt{grass-eater} \lor \texttt{mammal}, \neg\texttt{mammal} \lor \texttt{vertebrate}, \\ & \neg\texttt{vertebrate} \lor \texttt{animal}\}. \end{aligned}$$

For $S = \{\texttt{grass-eater}, \texttt{mammal}, \texttt{canine-teeth}\}$

We have that $B \models_S^3 \alpha$, hence $B \models \alpha$.

5. Refutation in S_1

Check whether $B \not\models \beta$, where $\beta = \neg\texttt{child} \vee \texttt{pensioner}$ and

$B = \{\neg\texttt{person} \vee \texttt{child} \vee \texttt{youngster} \vee \texttt{adult} \vee \texttt{senior},$

$\neg\texttt{adult} \vee \texttt{student} \vee \texttt{worker} \vee \texttt{unemployed},$

$\neg\texttt{pensioner} \vee \texttt{senior},\quad \neg\texttt{youngster} \vee \texttt{student} \vee \texttt{worker},$

$\neg\texttt{senior} \vee \texttt{pensioner} \vee \texttt{worker},\quad \neg\texttt{pensioner} \vee \neg\texttt{student},$

$\neg\texttt{student} \vee \texttt{child} \vee \texttt{youngster} \vee \texttt{adult},$

$\neg\texttt{pensioner} \vee \neg\texttt{worker}\}.$

For $S = \{\texttt{child}, \texttt{worker}, \texttt{pensioner}\}$.

5.1 S_1 simplification

- To decide whether $B \models_S^1 \alpha$:
  - If $p \notin S$, make $p, \neg p$ false in B
  - Obtain $B' \subseteq B$.
  - Apply classical SAT techniques to the resulting $B' \models \alpha$.
  - $B' \models \alpha$ iff $B \models_S^1 \alpha$, for $\alpha \in S$.

5.2 S_1 Example (cont)

Check whether $B \not\models \beta$, where $\beta = \neg$child $\vee$ pensioner and

B={ ¬person ∨ child ∨ youngster ∨ adult ∨ senior,
¬adult ∨ student ∨ worker ∨ unemployed,
¬pensioner ∨ senior, ¬youngster ∨ student ∨ worker,
¬senior ∨ pensioner ∨ worker, ¬pensioner ∨ ¬student,
¬student ∨ child ∨ youngster ∨ adult,
¬pensioner ∨ ¬worker}.

For $S = \{$child,worker, pensioner$\}$.

We have that $B \not\models^1_S \beta$, and hence $B \not\models \beta$.

6. Analysis of Cadoli & Schaerf's Method

- Good points of S_3:
  - S_3 approximates classical logic from below.
  - Nice, simple simplifications.
  - The set S defines a notion of relevance.
  - S_3 is paraconsistent: $p \wedge \neg p \not\models^3_S q \qquad \text{if } p \notin S$.
- Problems with S_3:
  - Clausal form only
  - Algorithm for simplification is not incremental.
  - Incremental method proposed, but no strategy to compute S is suggested.
  - No proof theory.

6.1 Problems with S_1

- S_1 does not approximate classical logic from above for:

$$\not\models^1_S p \vee \neg p, \qquad \text{if } p \notin S.$$

  (S_1 is paracomplete)

- S_1 cannot be extended to full propositional logic
- No strategy to compute S is suggested.
- $\models^1_S$ is not a local entailment:
  - To show that $B \not\models^1_S \alpha$, many irrelevant atoms have to be added to S, so that $v(B) = 1$.
  - No notion of relevance is given by S_1.

7. Next Topics

Part II: Approximate Theorem Proving:

- S_3 extended to the full propositional language.
- An incremental proof method for $S_3(S)$.
- A strategy to compute S.

Part III: Approximation of Classical Logic from Above

- The Family of Logics $\mathsf{s}_1(s)$.
- s_1 3-valued semantics for full propositional language.
- The notion of s_1-relevance.
- s_1-simplifications.