Parte III

Approximating Classical Logic "From Above"

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Contents

1. Approximation of Logics

- Interesting problems such as SAT and Theorem Proving have no known efficient algorithm.
  - Approximation of Logics is a possible way to face NP-complete and coNP-complete problems.
- Idealised agents are logically omniscient.
  - Real agents are limited.
  - Each step in an approximation models a limited agent.
- Approximations implicitly define a notion of relevance.

2. Schaerf & Cadoli's Proposal

- Restricted to Clausal Form (later NNF)
- Based on a context set S.
- If $p \in S$, p behaves classically

$$v(p) = 1 \quad \text{iff} \quad v(\neg p) = 0$$

- If $p \notin S$, p has a special behaviour:

$$\left.\begin{array}{lll} v(p) = 0 & \text{and} & v(\neg p) = 1 \\ v(p) = 1 & \text{and} & v(\neg p) = 1 \\ v(p) = 1 & \text{and} & v(\neg p) = 1 \end{array}\right\} S_3(S)$$

$$\left.\begin{array}{lll} v(p) = 0 & \text{and} & v(\neg p) = 0 \end{array}\right\} S_1(S)$$

2.1 Approximate Entailment

- Logics S_3 are useful to approximate Theorem Proving:

$$B \models_S^3 \alpha \Longrightarrow B \models \alpha$$

- Logics S_1 are useful to approximate "Theorem Disproving" or SAT:

$$B \not\models_S^1 \alpha \Longrightarrow B \not\models \alpha$$

- When $S = \mathcal{P}$, $S_1(S) = S_3(S) = CL$.

- **Theorem 1** *There exists an algorithm for deciding if* $B \models_S^3 \alpha$ *and deciding* $B \models_S^1 \alpha$ *which runs in* $O(|B|.|\alpha|.2^{|S|})$ *time.*

2.2 S_1 Example

Check whether $B \not\models \beta$, where $\beta = \neg\texttt{child} \vee \texttt{pensioner}$ and

$B = \{ \neg\texttt{person} \vee \texttt{child} \vee \texttt{youngster} \vee \texttt{adult} \vee \texttt{senior},$

$\neg\texttt{adult} \vee \texttt{student} \vee \texttt{worker} \vee \texttt{unemployed},$

$\neg\texttt{pensioner} \vee \texttt{senior}, \quad \neg\texttt{youngster} \vee \texttt{student} \vee \texttt{worker},$

$\neg\texttt{senior} \vee \texttt{pensioner} \vee \texttt{worker}, \quad \neg\texttt{pensioner} \vee \neg\texttt{student},$

$\neg\texttt{student} \vee \texttt{child} \vee \texttt{youngster} \vee \texttt{adult},$

$\neg\texttt{pensioner} \vee \neg\texttt{worker}\}.$

2.3 S_1 Example (solution)

Check whether $B \not\models \beta$, where $\beta = \neg \texttt{child} \vee \texttt{pensioner}$ and

$$
\begin{aligned}
B = \{ & \neg\texttt{person} \vee \texttt{child} \vee \texttt{youngster} \vee \texttt{adult} \vee \texttt{senior}, \\
& \neg\texttt{adult} \vee \texttt{student} \vee \texttt{worker} \vee \texttt{unemployed}, \\
& \neg\texttt{pensioner} \vee \texttt{senior}, \quad \neg\texttt{youngster} \vee \texttt{student} \vee \texttt{worker}, \\
& \neg\texttt{senior} \vee \texttt{pensioner} \vee \texttt{worker}, \quad \neg\texttt{pensioner} \vee \neg\texttt{student}, \\
& \neg\texttt{student} \vee \texttt{child} \vee \texttt{youngster} \vee \texttt{adult}, \\
& \neg\texttt{pensioner} \vee \neg\texttt{worker} \}.
\end{aligned}
$$

For $S = \{\texttt{child}, \texttt{worker}, \texttt{pensioner}\}$.

We have that $B \not\models^1_S \beta$, and hence $B \not\models \beta$.

3. The Notion of Approximation from Above

- We say that a family of parameterised logics $L(S)$ is an approximation of classical logic from above if for

$$\varnothing \subseteq S' \subseteq S'' \subseteq \ldots \subseteq S'^{n} \subseteq \mathcal{P}$$

we have that:

$$\models^L_{\varnothing} \quad \supseteq \quad \models^L_{S'} \quad \supseteq \quad \cdots \quad \supseteq \quad \models^L_{S'^{n}} \quad \supseteq \quad \models^L_{\mathcal{P}} = \models_{\mathsf{CL}}$$

Logics $L(S)$ have to contain *all* classical tautologies.

3.1 Problems with S_1

- S_1 does not approximate classical logic from above for:

$$\not\models^1_S p \vee \neg p, \qquad \text{if } p \notin S.$$

- S_1 cannot be extended to full propositional logic
- No strategy to compute S is suggested.
- $\models^1_S$ is not a local entailment:
  - To show that $B \not\models^1_S \alpha$, many irrelevant atoms have to be added to S, so that $v(B) = 1$.

4. The Family of Logics s_1

- $\mathsf{s}_1(s)$ is parameterised by the set $s \subseteq \mathcal{P}$.
- The language of s_1 is the full propositional language.
- $\mathsf{s}_1(s)$ has a 3-valued semantics.
- $v_s^1(\alpha) \subseteq \{0, 1\}$, but $v_s^1(\alpha) \neq \varnothing$.
- v_p: classical valuation v_p. For atomic symbols, v_s^1 extends v_p:

$$0 \in v_s^1(p) \Leftrightarrow v_p(p) = 0$$
$$1 \in v_s^1(p) \Leftrightarrow v_p(p) = 1 \;\; or \;\; p \notin s$$

4.1 Semantics of s_1

- We write $\alpha \in s$ iff $atoms(\alpha) \subseteq s$.
- Idea: If $\alpha \notin s$ then $1 \in v_s^1(\alpha)$.
- This the dual of S_1: If $\alpha \notin S$ then $v(\alpha) = 0$.
- The semantics of s_1 has to extend classical logic if we want it to be an approximation “from above”.

4.2 Definition of Semantics of s_1

Classical semantics for 0:

$$
\begin{aligned}
0 \in v_s^1(\neg\alpha) &\Leftrightarrow 1 \in v_s^1(\alpha) \\
0 \in v_s^1(\alpha \wedge \beta) &\Leftrightarrow 0 \in v_s^1(\alpha) \text{ or } 0 \in v_s^1(\beta) \\
0 \in v_s^1(\alpha \vee \beta) &\Leftrightarrow 0 \in v_s^1(\alpha) \text{ and } 0 \in v_s^1(\beta) \\
0 \in v_s^1(\alpha \to \beta) &\Leftrightarrow 1 \in v_s^1(\alpha) \text{ and } 0 \in v_s^1(\beta)
\end{aligned}
$$

"Extended" classical semantics for 1:

$$
\begin{aligned}
1 \in v_s^1(\neg\alpha) &\Leftrightarrow 0 \in v_s^1(\alpha) && \text{or } \neg\alpha \notin s \\
1 \in v_s^1(\alpha \wedge \beta) &\Leftrightarrow 1 \in v_s^1(\alpha) \text{ and } 1 \in v_s^1(\beta) && \text{or } \alpha \wedge \beta \notin s \\
1 \in v_s^1(\alpha \vee \beta) &\Leftrightarrow 1 \in v_s^1(\alpha) \text{ or } 1 \in v_s^1(\beta) && \text{or } \alpha \vee \beta \notin s \\
1 \in v_s^1(\alpha \to \beta) &\Leftrightarrow 0 \in v_s^1(\alpha) \text{ or } 1 \in v_s^1(\beta) && \text{or } \alpha \to \beta \notin s
\end{aligned}
$$

4.3 Properties of s_1

- $v_s^1(\alpha) \neq \varnothing$.
- If $\alpha \notin s$ then $1 \in v_s^1(\alpha)$.
- Let v_c classically extend v_p. Then, $v_c(\alpha) \in v_s^1(\alpha)$.
- If $\alpha \in s$, $v_s^1(\alpha) = \{v_c(\alpha)\}$.
- If $s \subseteq s'$ then $v_s^1(\alpha) \supseteq v_{s'}^1(\alpha)$.

5. s_1 Entailment

- We want to extend $B \models \alpha$
- If $v_s^1(\alpha) = \{1\}$ then we say that α is strictly satisfied by v_s^1.
- If $1 \in v_s^1(\alpha)$ then we say that α is relaxedly satisfied by v_s^1.
- Properties:
  - α is strictly satisfiable $\Longrightarrow$ α is classically satisfiable.
  - α is classically satisfiable $\Longrightarrow$ α is relaxedly satisfiable.
- <u>Definition</u>: $B \models_s^1 \alpha$ iff every v_s^1 that strictly satisfies all $\beta_i \in B$ also relaxedly satisfies α.
- That is, whenever $v_s^1(B) = \{1\}$ then $1 \in v_s^1(\alpha)$.

5.1 Properties of s_1 Entailment

- $B \models^1_\varnothing \alpha$, for every $\alpha \in \mathcal{L}$.
- $\models^1_\mathcal{P} = \models_{\mathsf{CL}}$
- If $s \subseteq s'$, $\models^1_s \supseteq \models^1_{s'}$.
- It follows that the family of s_1-logics approximates classical entailment from above, that is:

$$\models^1_\varnothing \supseteq \models^1_{s'} \supseteq \ \dots \ \supseteq \models^1_{s'^n} \supseteq \models^1_\mathcal{P} = \models_{\mathsf{CL}}$$

for

$$\varnothing \subseteq s' \subseteq s'' \subseteq \ldots \subseteq s'^n \subseteq \mathcal{P}$$

5.2 Example Revisited in s_1

Check whether $B \not\models \beta$, where $\beta = \neg\texttt{child} \vee \texttt{pensioner}$ and

$$
\begin{aligned}
B = \{ & \neg\texttt{person} \vee \texttt{child} \vee \texttt{youngster} \vee \texttt{adult} \vee \texttt{senior}, \\
& \neg\texttt{adult} \vee \texttt{student} \vee \texttt{worker} \vee \texttt{unemployed}, \\
& \neg\texttt{pensioner} \vee \texttt{senior}, \quad \neg\texttt{youngster} \vee \texttt{student} \vee \texttt{worker}, \\
& \neg\texttt{senior} \vee \texttt{pensioner} \vee \texttt{worker}, \quad \neg\texttt{pensioner} \vee \neg\texttt{student}, \\
& \neg\texttt{student} \vee \texttt{child} \vee \texttt{youngster} \vee \texttt{adult}, \\
& \neg\texttt{pensioner} \vee \neg\texttt{worker} \}.
\end{aligned}
$$

For $s = \{\texttt{child}, \texttt{pensioner}\}$ ($\texttt{worker} \notin s$)

5.3 Example Revisited (cont.)

In this case, it suffices to examine just the following.

B={ ¬person ∨ child ∨ youngster ∨ adult ∨ senior,

¬adult ∨ student ∨ worker ∨ unemployed,

¬pensioner ∨ senior, ¬youngster ∨ student ∨ worker,

¬senior ∨ pensioner ∨ worker, **¬pensioner** ∨ ¬student,

¬student ∨ child ∨ youngster ∨ adult,

¬pensioner ∨ ¬worker}.

Take $v_p(\texttt{pensioner}) = 0$ and $v_p(p) = 1$ otherwise.

The corresponding v_s^1 gives: $v_s^1(B) = \{1\}$ and $v_s^1(\beta) = \{0\}$, so $B \not\models_s^1 \beta$.

Hence, $B \not\models \beta$.

5.4 Locality and Relevance

Consider the following example, representing beliefs about a young student.

$$B = \{\texttt{student}, \texttt{student} \rightarrow \texttt{young}, \texttt{young} \rightarrow \neg\texttt{pensioner}, \texttt{worker}, \texttt{worker} \rightarrow \neg\texttt{pensioner}, \texttt{blue-eyes}, \texttt{likes-dancing}, \texttt{six-feet-tall}\}.$$

We want to know whether $B \models \texttt{pensioner}$.

- In S_1, S must contain at least one atom of each clause, even irrelevant ones such as `likes-dancing`.
- In s_1, with s=$\{\texttt{pensioner}\}$, fix $v_p(\texttt{pensioner}) = 0$ and $v_p(p) = 1$ otherwise.
- This makes $B \not\models \texttt{pensioner}$.

6. Tableaux for s_1

- KE tableaux deal with T- and F-signed formulas: $T\ \alpha$ and $F\ \alpha$.
- KEs_1-Tableaux extend classical KE-tableaux.
- KEs_1 deals with T and F signs, and also with two new signs: 1 and 0.

$$\begin{array}{lcl} T\ \alpha & \Longrightarrow & v(A) = \{1\} \\ F\ \alpha & \Longrightarrow & v(A) = \{0\} \\ 1\ \alpha & \Longrightarrow & 1 \in v(A) \\ 0\ \alpha & \Longrightarrow & 0 \in v(A) \end{array}$$

These four signs are not mutually exclusive.

Branching Rules and Promotion Rules

Two versions of the Principle of Bivalence:

$$\begin{array}{c} \diagup \quad \diagdown \\ 1\,\alpha \qquad F\,\alpha \end{array} \quad (\mathrm{PB}_{1F}) \qquad\qquad \begin{array}{c} \diagup \quad \diagdown \\ T\,\alpha \qquad 0\,\alpha \end{array} \quad (\mathrm{PB}_{T0})$$

Promotion Rules

$$\frac{1\,\alpha}{T\,\alpha}\,\alpha \in s \qquad \frac{0\,\alpha}{F\,\alpha}\,\alpha \in s$$

Connective Rules

$\dfrac{\begin{array}{c}T\ \alpha\to\beta\\ 0\ \beta\end{array}}{F\ \alpha}$	$\dfrac{F\ \alpha\to\beta}{\begin{array}{c}T\ \alpha\\ F\ \beta\end{array}}$	$\dfrac{\begin{array}{c}1\ \alpha\to\beta\\ T\ \alpha\end{array}}{1\ \beta}$	$\dfrac{0\ \alpha\to\beta}{\begin{array}{c}1\ \alpha\\ 0\ \beta\end{array}}$
$\dfrac{T\ \alpha\land\beta}{\begin{array}{c}T\ \alpha\\ T\ \beta\end{array}}$	$\dfrac{\begin{array}{c}F\ \alpha\land\beta\\ 1\ \alpha\end{array}}{F\ \beta}$	$\dfrac{1\ \alpha\land\beta}{\begin{array}{c}1\ \alpha\\ 1\ \beta\end{array}}$	$\dfrac{\begin{array}{c}0\ \alpha\land\beta\\ T\ \alpha\end{array}}{0\ \beta}$
$\dfrac{\begin{array}{c}T\ \alpha\lor\beta\\ 0\ \alpha\end{array}}{T\ \beta}$	$\dfrac{F\ \alpha\lor\beta}{\begin{array}{c}F\ \alpha\\ F\ \beta\end{array}}$	$\dfrac{\begin{array}{c}1\ \alpha\lor\beta\\ F\ \alpha\end{array}}{1\ \beta}$	$\dfrac{0\ \alpha\lor\beta}{\begin{array}{c}0\ \alpha\\ 0\ \beta\end{array}}$
$\dfrac{T\ \neg\alpha}{F\ \alpha}$	$\dfrac{F\ \neg\alpha}{T\ \alpha}$	$\dfrac{1\ \neg\alpha}{0\ \alpha}$	$\dfrac{0\ \neg\alpha}{1\ \alpha}$

Strong and Defeasible Closings

Strong Closings

$$\begin{array}{ccc} T\,\alpha & 1\,\alpha & 0\,\alpha \\ F\,\alpha & F\,\alpha & T\,\alpha \\ \times & \times & \times \end{array}$$

Defeasible Closing

$$\begin{array}{c} F\,\alpha \quad \alpha \notin s \\ - \end{array}$$

6.1 Properties of KE_{S_1}

- All classical KE connective rules are derivable.
- Soundness and completeness:

$$B \vdash_s^1 \beta \textit{ iff } B \models_s^1 \beta.$$

6.2 KE_{S_1}-Tableau: an Example

We want to check whether $p \rightarrow q, q \vdash_s^1 p$.

1.	$T\ p \rightarrow q$	by hypothesis, $s = \varnothing$
2.	$T\ q$	by hypothesis
3.	$F\ p$	by hypothesis
4.	–	defeasible closure from 3.

6.3 KE_{S_1}-Tableau: an Example

We want to check whether $p \to q, q \vdash_s^1 p$.

1.	$T\ p \to q$	by hypothesis, $s = \varnothing$
2.	$T\ q$	by hypothesis
3.	$F\ p$	by hypothesis
4.	–	defeasible closure from 3. Reopen with $s = \{p\}$
5.	$0\ q$ / $T\ q$	PB_{0T}
6.	$F\ p$	by rule $(0 \to)$ on 1 and 5

As usual, an open branch gives us a valuation that refutes the initial sequent. Right branch gives us $v_s^1(q) = \{1\}, v_s^1(p) = \{0\}$, which is a classical valuation.

Conclusions and the Future

- s_1-entailment is an approximation from above.
- It works for full propositional logic.
- s_1 has the locality property and defines a relevance notion.
- KEs_1, an incremental proof method for s_1.
- Future work:
  - Complexity of s_1-SAT.
  - Applications to belief revision and other logics.