Real Number Calculations and Interval Analysis in PVS *

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Universidade de Brasilia – Abril 2006

* Based on "Real Number Calculations" (TPHOLs 2005) by C. Muñoz and D. Lester, and "Guaranteed Proofs Using Interval Arithmetic" (ARITH 17) by M. Daumas, G. Melquiond, and C. Muñoz.

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International Space Station

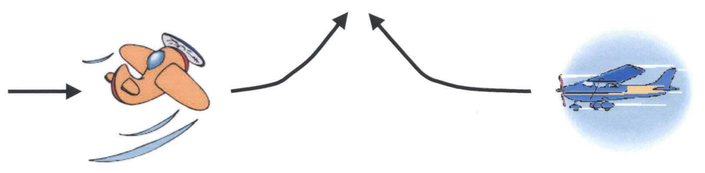
ISS012E22281

Marco Pontes: Crew member of Expedition 13.

Conflict Detection and Resolution

A NASA Application

How to avoid these situations ?



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Requirements

- Distributed: Responsibility for separation resides on the pilot.
- Pair-wise: Two aircraft (ownship and intruder)
- Tactical: Detection and resolution are based on state information (as opposed to intent information).
- Implicit coordination: No information is exchanged between the aircraft (but aircraft are aware of traffic position and velocity).
- Independent coordination: Resolution are repulsive, but cooperation is not required to keep minimum separation.

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KB3D

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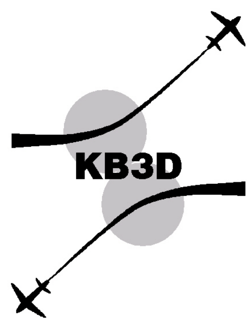
KB3D is an algorithm designed and verified by the Formal Methods group at NIA-NASA Langley.



Conflict Detection: Problem

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Given the relative 3-D position $\vec{s} = (s_x, s_y, s_z)$ and velocity vector $\vec{v} = (v_x, v_y, v_z)$ of the *ownship* with respect to an *intruder* aircraft, define

$\texttt{cd3d}(s_x, s_y, s_z, v_x, v_y, v_z)\texttt{:bool}$

such that
$\texttt{cd3d}(s_x, s_y, s_z, v_x, v_y, v_z) \equiv$
"There is a predicted conflict within a lookahead time T" $\equiv$

$\exists_{0 \le t < T} (s_x + t v_x)^2 + (s_y + t v_y)^2 < D^2$ and
$(s_z + t v_z)^2 < H^2$

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Conflict Detection (in PVS)

```
cd3d(s_x,s_y,s_z,v_x,v_y,v_z:real) : bool =
  IF  v_x=0 && v_y=0 && s_x^2+s_y^2<D^2 THEN
    s_z^2<H^2 || (v_z s_z<0 && -H<sign(v_z)(T v_z+s_z))
  ELSE
    LET d = 2 s_x v_x s_y v_y+D^2 (v_x^2+v_y^2) - (s_x^2 v_y^2+s_y^2 v_x^2) IN
    IF d>0 THEN
      LET a = v_x^2+v_y^2 IN
      LET b = s_x v_x+s_y v_y IN
      IF v_z = 0 THEN
        s_z^2<H^2 && (D^2>s_x^2+s_y^2 || b≤0) && (d>(a T+b)^2 || a T+b≥0)
      ELSE
        LET t_1 = (-sign(v_z) H-s_z)/v_z IN
        LET t_2 = (sign(v_z) H-s_z)/v_z IN
        (d>(a t_2+b)^2 || a t_2+b≥0) && (d>(a t_1+b)^2 || a t_1+b≤0) &&
        (D^2>s_x^2+s_y^2 || b≤0) && (d>(a T+b)^2 || a T+b≥0) &&
        t_1<T && t_2>0
        ENDIF
    ELSE FALSE
    ENDIF
  ENDIF
```


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Conflict Detection: Verification

- Correctness: THEOREM
  $\forall\ \vec{s}, \vec{v}.$
  $\mathtt{cd3d}(\vec{s}, \vec{v}) \implies$
  $\exists_{0 \le t < T}.\ (s_x + tv_x)^2 + (s_y + tv_y)^2 < D^2\ \wedge\ (s_z + tv_z)^2 < H^2.$

- Completeness: THEOREM
  $\forall\ \vec{s}, \vec{v}.$
  $(\exists_{0 \le t < T}.\ (s_x + tv_x)^2 + (s_y + tv_y)^2 < D^2\ \wedge\ (s_z + tv_z)^2 < H^2) \implies$
  $\mathtt{cd3d}(\vec{s}, \vec{v}).$

- Both theorems were formally verified using a mechanical theorem prover.

Example 1.*

- The turn rate of an aircraft flying at v knots and with a bank angle of ϕ is given by the formula

$$\dot{\theta}(\phi, v) = g \tan(\phi)/v,$$

where $g = 9.8$ m/s^2.

- What is $\dot{\theta}(35^o, 250knots)$?

*Taken from *Formal verification of conflict detection algorithms*, C. Muñoz, R. Butler, V. Carreño, and G. Dowek, 2003.

In Java ...

```
class thetadot {

  static final double g = 9.8;

  static double thetadot(double phi,double v) {
    return 1000*g*Math.tan(phi*Math.PI/180.0)/(514*v);
  }

  public static void main(String argv[]) {
    double phi = 35;
    double v   = 250;
    System.out.println("thetadot("+phi+","+v+") = "+
                       thetadot(phi,v));
  }
}
```

```
$ java thetadot
thetadot(35.0,250.0) = 0.05340104182455374
```


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In PVS . . .

```
thetadot : THEORY
BEGIN
  g : posreal = 9.8

  thetadot (phi:real,v:posreal) : real =
    1000*g*tan(phi*pi/180)/(514*v)

  phi : posreal = 35
  v   : posreal = 250

  thetadot_35_250 : axiom
    0.053 <= thetadot(phi,v) AND
    thetadot(phi,v) <= 0.054

END thetadot
```


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Example 2.[¶]

- Given the following definition:

$$
\begin{aligned}
a_0 &= 11/2 \\
a_1 &= 61/11 \\
a_{n+2} &= 111 - \frac{1130 - 3000/a_n}{a_{n+1}}
\end{aligned}
$$

- What is a_{20} ?

[¶] Taken from *Arithmétique des ordinateurs, J.-M. Muller, 1989.*

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In Java . . .

```
class mya {

  static double a(int n) {
    if (n==0)
      return 11/2.0;
    if (n==1)
      return 61/11.0;
    return 111 - (1130 - 3000/a(n-2))/a(n-1);
  }

  public static void main(String[] argv) {
    for (int i=0;i<=20;i++)
      System.out.println("a("+i+") = "+a(i));
  }
}
```

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$a_0 \ldots a_{20}$

```
$ java mya
a(0)  = 5.5
a(2)  = 5.5901639344262435
a(4)  = 5.674648620514802
a(6)  = 5.74912092113604
a(8)  = 5.81131466923334
a(10) = 5.861078484508624
a(12) = 5.935956716634138
a(14) = 15.413043180845833
a(16) = 97.13715118465481
a(18) = 99.98953968869486
a(20) = 99.99996275956511
```

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In PVS ...

```
mya : THEORY
BEGIN

    a(n:nat) : RECURSIVE posreal =
      if    n = 0 then 11/2
      elsif n = 1 then 61/11
      else  111 - (1130 - 3000/a(n-2))/a(n-1)
      endif
      MEASURE n

    a20_axiom : axiom
      99 <= a(20) AND a(20) <= 100

    a20_lemma : lemma
      a(20) <= 6
    %|- a20_lemma : PROOF (eval-formula) QED

END mva
```

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How to Proof this Simple Lemma ?

```
    g   : posreal = 9.8
    phi : posreal = 35
    v   : posreal = 250

    td_35_250 : lemma
      0.053 <= 1000*g*tan(phi*pi/180)/(514*v)

%|- td_35_250 : PROOF ( ... ) QED
```

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A Simple Lemma ?

- There are no variables. How hard could this be ?

```
  tr_35 : lemma
    3*pi/180 <= tr(35*pi/180)
%|- tr_35 : PROOF
%|- First note that that tan is increasing.
%|- Furthermore, π/6  <  35π/180. Moreover, ...
%|- QED
```


A Mechanical Proof

- Assume $\underline{\pi}, \overline{\pi}, \underline{\tan}, \overline{\tan}$, lower and upper bounds of π and tan, respectively.
- Goal:

$$\vdash \; 3\pi/180 \; \leq \; v\tan(35\pi/180)/g.$$

- *Proof*:
  1. $\vdash \; 3\pi/180 \; \leq \; 3\overline{\pi}/180$, because $\overline{\pi}$ is an upper bound.
  2. $\vdash \; v\underline{\tan}(35\underline{\pi}/180)/g \; \leq \; v\tan(35\pi/180)/g$, because $\underline{\tan}$ and $\underline{\pi}$ are lower bounds.
  3. $\vdash \; 3\overline{\pi}/180 \; \leq \; v\underline{\tan}(35\underline{\pi}/180)/g$, by simple calculation.

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Issues

1. What are the requirements for $\underline{f}$ and $\overline{f}$?
2. How to use $\underline{f}$ and $\overline{f}$ in a systematic way?
3. How to automate those proofs in PVS ?

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Issue 1: $\underline{f}$ and $\overline{f}$

The functions $\underline{f} : (\mathbb{R}, \mathbb{N}) \rightarrow \mathbb{R}$ and $\overline{f} : (\mathbb{R}, \mathbb{N}) \rightarrow \mathbb{R}$ are closed under $\mathbb{Q}$ such that

$$\begin{aligned} \underline{f}(x,n) \;\leq\; & f(x) \;\leq\; \overline{f}(x,n), & (1)\\ \underline{f}(x,n) \;\leq\; & \underline{f}(x,n+1), & (2)\\ \overline{f}(x,n+1) \;\leq\; & \overline{f}(x,n), & (3)\\ \lim_{n\rightarrow\infty} \underline{f}(x,n) \;=\; & f(x) \;=\; \lim_{n\rightarrow\infty} \overline{f}(x,n), & (4) \end{aligned}$$

where n is an approximation parameter.

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Solution Issue 1: Rational Lower Bounds

We have defined $\underline{f}$ and $\overline{f}$ for $f \in \{\sin, \cos, \tan, \pi, \mathrm{atan}, \sqrt{}, \exp, \ln\}$ and formally verified (in PVS) that they satisfy (1)-(4).

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Sine, Cosine

$$\begin{aligned}
\underline{\sin}(x,n) &= \sum_{i=1}^{2n} (-1)^{i-1} \frac{x^{2i-1}}{(2i-1)!}, \\
\overline{\sin}(x,n) &= \sum_{i=1}^{2n+1} (-1)^{i-1} \frac{x^{2i-1}}{(2i-1)!}, \\
\underline{\cos}(x,n) &= 1 + \sum_{i=1}^{2n+1} (-1)^{i} \frac{x^{2i}}{(2i)!}, \\
\overline{\cos}(x,n) &= 1 + \sum_{i=1}^{2(n+1)} (-1)^{i} \frac{x^{2i}}{(2i)!}.
\end{aligned}$$

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Logarithm $(1 < x \leq 2)$

For $-1 < x \leq 1$, we use the alternating series for natural logarithm:

$$\ln(x+1) \;=\; \sum_{i=1}^{\infty}(-1)^{i+1}\frac{x^i}{i}.$$

Therefore, we define

$$\underline{\ln}(x,n) \;=\; \sum_{i=1}^{2n}(-1)^{i+1}\frac{(x-1)^i}{i}, \quad \text{if } 1 < x \leq 2,$$

$$\overline{\ln}(x,n) \;=\; \sum_{i=1}^{2n+1}(-1)^{i+1}\frac{(x-1)^i}{i}, \quad \text{if } 1 < x \leq 2.$$

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Logarithm $(0 < x \leq 1)$

Using properties of the natural logarithm function, we obtain

$$\underline{\ln}(1,n) \;=\; \overline{\ln}(1,n) \;=\; 0,$$

$$\underline{\ln}(x,n) \;=\; -\underline{\ln}(\frac{1}{x},n), \quad \text{if } 0 < x < 1,$$

$$\overline{\ln}(x,n) \;=\; -\overline{\ln}(\frac{1}{x},n), \quad \text{if } 0 < x < 1.$$

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Logarithm $(2 < x)$

We observe that

$$\exists(m{:}\mathbb{N}, y{:}(1\ldots2]) : \ln(x) = m\ln(2) + \ln(y).$$

Hence,

$$\begin{array}{rcll} \underline{\ln}(x,n) & = & m\ \underline{\ln}(2,n) + \underline{\ln}(y,n), & \text{if } x > 2, \\ \overline{\ln}(x,n) & = & m\ \overline{\ln}(2,n) + \overline{\ln}(y,n), & \text{if } x > 2. \end{array}$$

Furthermore, $\surd$, tan, π, atan, exp, ...

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Issue 2: $\overline{f}$ or $\underline{f}$?

Note that

$$y + f(x) \quad \le \quad y + \overline{f}(x,n),$$

but

$$y - f(x) \quad \le \quad y - \underline{f}(x,n).$$

In general,

$$\begin{array}{rcll} k \times f(x) & \le & k \times F(x,n), & \text{where} \\ F & = & \overline{f} & \text{if } k \ge 0, \\ F & = & \underline{f} & \text{otherwise.} \end{array}$$

To chose the appropriate F, we first have to decide whether k is positive or negative.

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A Difficult Decision Problem

$$\frac{k}{f(x)} \leq \frac{k}{F(x,n)}.$$

Which bound $F = \overline{f}$ or $F = \underline{f}$ is appropriate in this case ?

Solution Issue 2: Rational Interval Arithmetic

- Let $\underline{\mathbf{x}}, \overline{\mathbf{x}}$ be in $\mathbb{Q}$,

$$\mathbf{x} \;=\; [\underline{\mathbf{x}}, \overline{\mathbf{x}}] \;=\; \{x \mid \underline{\mathbf{x}} \leq x \leq \overline{\mathbf{x}}\}.$$

- Interval basic operations are defined such that they satisfy the inclusion property: If $x \in \mathbf{x}$ and $y \in \mathbf{y}$ then

$$\begin{aligned} x \otimes y &\in \mathbf{x} \otimes \mathbf{y}, \quad \text{where } \otimes \in \{+, -, \times, \div\}, \\ -x &\in -\mathbf{x}, \\ |x| &\in |\mathbf{x}|, \quad \text{and} \\ x^n &\in \mathbf{x}^n. \end{aligned}$$

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From Real Functions to Interval Functions

- For each f, a parametric interval function $[\mathbf{f}]_n$ is defined as follows:

$$
\begin{array}{lcll}
[\mathbf{f}(\mathbf{x})]_n & = & [\underline{f}(\underline{\mathbf{x}}, n), \overline{f}(\overline{\mathbf{x}}, n)], & \text{if } f \text{ is increasing,} \\
[\mathbf{f}(\mathbf{x})]_n & = & [\underline{f}(\overline{\mathbf{x}}, n), \overline{f}(\underline{\mathbf{x}}, n)], & \text{if } f \text{ is decreasing.}
\end{array}
$$

- If f is neither increasing nor decreasing, e.g., sin and cos, $[\mathbf{f}]_n$ is defined by case analysis on increasing and decreasing ranges.

Extended Inclusion Property

- If $x \in \mathbf{x}$, then $f(x) \in [\mathbf{f}(\mathbf{x})]_n$, for all $n \in \mathbb{N}$.

- We have defined $[\mathbf{f}(\mathbf{x})]_n$ for $f \in \{\text{sin}, \text{cos}, \text{tan}, \pi, \text{atan}, \sqrt{}, \text{exp}, \text{ln}\}$ and formally verified (in PVS) the inclusion property for each interval operation.

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Inclusion Theorem

- Let e be a real expression on variables $x_1, \ldots x_m$, and let $\mathbf{x_1}, \ldots, \mathbf{x_m}$ be interval values such that $x_i \in \mathbf{x_i}$, for $1 \leq i \leq m$, then

$$e(x_1, \ldots, x_m) \in [\mathbf{e}(\mathbf{x_1}, \ldots, \mathbf{x_m})]_n,$$

  where $[e]_n$ is the interval expression corresponding to e.
- *Proof*: Structural induction on e.

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General (Incomplete) Method

- Goal:

$$x_1 \in \mathbf{x_1}, \ldots, x_m \in \mathbf{x_m} \ \vdash \ e(x_1, \ldots, x_m) \ \diamond \ k,$$

  where $\diamond \in \{<, \leq, >, \geq\}$.
- *Proof*:
  1. Derive:

$$x_1 \in \mathbf{x_1}, \ldots, x_m \in \mathbf{x_m} \ \vdash \ e(x_1, \ldots, x_m) \in [\mathbf{e}(\mathbf{x_1}, \ldots, \mathbf{x_m})]_n,$$

     for a given n, using the Inclusion Theorem.
  2. Check by simple calculation:

$$\vdash \ [\mathbf{e}(\mathbf{x_1}, \ldots, \mathbf{x_m})]_n \ \diamond \ k.$$

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Issue 3: Automation

The Sources of Incompleteness:

- Which n ?
- Interval arithmetic is sub-distributive:

$$\mathbf{x} \times (\mathbf{y} + \mathbf{z}) \subseteq \mathbf{x} \times \mathbf{y} + \mathbf{x} \times \mathbf{z}.$$

  Decorrelation effect:
  - $\mathbf{x} - \mathbf{x}$ is not necessarily $\mathbf{0}$,
  - $\mathbf{x} \div \mathbf{x}$ is not necessarily $\mathbf{1}$,
  - $\mathbf{x} \geq 0$ and $\mathbf{x} \leq 0$ may be both false.
- Interval computations yield correct approximations, but not necessarily good ones.

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Solution Issue 3: Pragmatism

- Which n ? A configurable parameter with a default value, e.g., 3 suffices in most of our applications.
- Implementation of rewriting strategies to rearrange and simplify expressions: factorized expressions have fewer decorrelation.

Real Number Calculations and Interval Analysis in PVS [a]

[a] Based on "Real Number Calculations" (TPHOLs 2005) by C. Muñoz and D. Lester, and "Guaranteed Proofs Using Interval Arithmetic" (ARITH 17) by M. Daumas, G. Melquiond, and C. Muñoz.

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Interval Splitting

The approximation error of the union of the parts is less than the approximation error of the whole:

Let $\mathbf{x} = \bigcup_{1 \leq i \leq n} \mathbf{x_i}$,

$$\frac{\forall_{1 \leq i \leq n} : \; x \in \mathbf{x_i} \; \vdash \; e(x) \in \mathbf{z}}{x \in \mathbf{x} \; \vdash \; e(x) \in \mathbf{z}}$$

Remark: $\mathbf{z} \subseteq [\mathbf{e}(\mathbf{x})]_n$.

Implementation: An interval $\mathbf{x}$ is evenly split using a user-provided parameter.

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Taylor's Theorem on Intervals

The derivative of $f = \frac{df}{dx}$ has one degree less of decorrelation than f.

$$\frac{a \in \mathbf{x} \; \vdash \; \frac{d^i \; f}{dx^i}(a) \in \mathbf{x}_i, \quad \text{for } 0 \leq i < n, \quad \forall_y : \; y \in \mathbf{x} \; \vdash \; \frac{d^n \; f}{dx^n}(y) \in \mathbf{x}_n}{x \in \mathbf{x} \; \vdash \; f(x) \in \Sigma_{k=0}^{n}(\mathbf{x}_k \times (\mathbf{x} - a)^k)/k!}$$

Remark: $\Sigma_{k=0}^{n}(\mathbf{x}_k \times (\mathbf{x} - a)^k)/k! \subseteq [\mathbf{f}(\mathbf{x})]_n$.

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Taylor's Implementation

- The element a is the midpoint of $\mathbf{x}$.
- $\mathbf{x_i}$ is the interval expression corresponding to $\frac{d^i\ f}{dx^i}(a)$, for $0 \leq i < n$.
- $\mathbf{x_n}$ is the interval expression corresponding to $\frac{d^n\ f}{dx^n}(y)$.

PVS Implementation Issues

- Representation of rational numbers: PVS built-in reals.
  + No special syntax for real expressions.
  - The Inclusion Theorem has to be discharged for every instance.
- Efficient calculations via computational reflection: Ground rational expressions are evaluated in Lisp.

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The Interval Package (for PVS)

- Publicly available: `http://research.nianet.org/~munoz/Interval`.
- Several strategies, notably
  - `sharp`, which automatically discharges the Inclusion Theorem.
  - `numerical`, which implements splitting.
  - `taylor`, which implements Taylor's technique.

Statistics

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- 306 lemmas, 10.000 lines of proofs, 1.000 lines of strategy code.
- These numbers do not include the bounding functions which are part of the NASA PVS Library: `http://shemesh.larc.nasa.gov/fm/ftp/larc/PVS-library/pvsl`

Examples

```
g : posreal = 9.8
v : posreal = 250×0.514

tr35: LEMMA 3×π/180 ≤ g×tan(35×π/180)/v
%|- tr35: PROOF (numerical) QED

G(x|x < 1): real = 3×x/2 - ln(1-x)

A_and_S : lemma
  let x = 0.5828 in
    G(x) > 0
%|- A_and_S : PROOF (numerical :defs "G") QED
```

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More Examples

```
% A fair approximation:

{-1}  x ## [| 0, 1 |]
  |-------
{1}   x * (1 - x) ## [| 0, 1 |]

Rule? (numerical :vars "x")
Q.E.D.

% A better approximation (via splitting):

{-1}  x ## [| 0, 1 |]
  |-------
{1}   x * (1 - x) ## [| 0, 9 / 32 |]

Rule? (numerical :vars ("x" 16))
Q.E.D.
```

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More Examples

```
% The best approximation (via Taylor's technique):

X : var Interval
x : var inInterval([|0,1|])

F(X)   : Interval = X*(1-X)
DF(X)  : Interval = 1 - 2*X
D2F(X) : Interval = [| -2 |]

ftaylor : LEMMA
  x*(1-x) ## Taylor2[[|0,1|]](F,DF,D2F)
%|- ftaylor : PROOF (taylor) QED

best : LEMMA
  x*(1-x) ## [| 0,1/4 |]
%|- best : PROOF (numerical :vars "x"
%|-                         :taylor "ftaylor") QED
```


Final Remarks

- All the results are written and formally verified in PVS.
- Strategies provide a pragmatic approach to real exact arithmetic.
- Applications: Verification of aerospace applications.
- Future work: Fewer decorrelation, higher accuracy, floating point numbers, ...

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