Introduction to Linear Logic

Beniamino Accattoli

INRIA and LIX (École Polytechnique)

Outline

Outline

Quotation

From ***A taste of Linear Logic*** of Philip Wadler:

- Some of the best things in life are ***free***; and some ***are not***.
- Truth is ***free***.
- You may use a proof of a theorem ***as many times as you wish***.
- Food, on the other hand, has a ***cost***.
- Having baked a cake, you may eat it ***only once***.
- If traditional logic is about ***truth***, then

  Linear Logic is about food

Informally 1

Classical logic deals with stable truths:

if A and $A \Rightarrow B$
then B
but A still holds

Example:

1. A = 'Tomorrow is the 1st october'.
2. B = 'John will go to the beach'.
3. $A \Rightarrow B$ = 'If tomorrow is the 1st october then John will go to the beach'.

So if ***tomorrow is the 1st october***, then ***John will go to the beach***,

But of course tomorrow ***will still be the 1st october***.

Informally 2

But with money, or food, that implication is wrong:

1. A = 'John has (only) 5 euros'.
2. B = 'John has a packet of cigarettes'.
3. $A \Rightarrow B$ = 'for his 5 euros John gets a packet of cigarettes'.

If ***John buys the cigarettes*** then he still has that 5 euros!

The world described by classical logic is quite a peculiar world...

Informally 3

In ***Linear Logic***:

- Implication ***consumes hypotesis***, to ***produce conclusions***.
- Linear implications are ***actions*** (they can represent the concept of action as found in AI).
- But LL is more than just consuming hypothesis.
- There is also a way to talk about eternal truths.
- LL is ***not*** a new type of logic, it ***refines*** classical logic.
- There are ***two*** conjunctions, $\otimes$ and $\oplus$, ***two*** disjunctions, $⅋$ and $\oplus$, and also two ***modalities***, $!$ and $?$.
- The propositional part of the logic becomes ***much more expressive*** than before...

Outline

Rules vs Axioms

Deductions are ***reflexive***, $A \Rightarrow A$ for every A, and ***transitive***:

$$\textit{if } A \Rightarrow B \textit{ and } B \Rightarrow C$$

$$\textit{then } A \Rightarrow C$$

Two ***opposite*** kinds of ***formal proof system***:

1. ***Hilbert***:
   - ***Axioms***: for ***every*** connective,
   - ***Rules***: ***only*** transitivity.
2. ***Gentzen***:
   - ***Axioms***: ***only*** one (scheme), reflexivity,
   - ***Rules***: for ***every*** connective.

Hilbert-style deduction systems

Inter-definibility of connectives $\Rightarrow$ ***3 axiom schemes***:

$$A \Rightarrow (B \Rightarrow A)$$

$$(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$$

$$(\neg A \Rightarrow \neg B) \Rightarrow (B \Rightarrow A)$$

And the ***transitivity rule***, the ***modus ponens***

$$\frac{A \quad A \Rightarrow B}{B}$$

Remark: A is in the premises but ***not*** in the conclusion, so if we want to build a proof of B, knowing just B, we have to ***guess*** A...

Gentzen's sequent calculus

Sequents are syntactical objects of the shape:

$$\Gamma \vdash \Delta$$

where Γ and Δ are ***sets*** of formulas.

- To be read:

  *if **all** formulas in Γ are true then **one** of the formulas in Δ is true*'

- The turnstile '$\vdash$' is a ***meta***-notation for 'implies', or 'proves'.
- Axioms ***only*** for the ***reflexivity of deduction***:

$$A \vdash A$$

  for no matter what formula A.

Rules

Identity group:

$$\frac{}{A \vdash A}(Ax) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}(cut)$$

Logical Group:

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}(l\wedge) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'}(r\wedge)$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'}(l\vee) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}(r\vee)$$

Remark: commas are connectives!

Negation and truth values:

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}(l\neg) \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}(r\neg) \qquad \frac{}{\bot \vdash}(False) \qquad \frac{}{\vdash \top}(True)$$

Structural Rules

A sequent is composed by two ***sets of formulas***.

Sets are formalized via the ***structural rules***:

- ***Contraction***:

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}(lContr) \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}(rContr)$$

- ***Weakening***:

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}(lWeak) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}(rWeak)$$

- ***Exchange***:

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, B, A \vdash \Delta}(lExc) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash B, A, \Delta}(rExc)$$

Outline

Two words about negation

- The rules about negation are very ***counter-intuitive***:

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}(l\neg) \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}(r\neg)$$

  premises become ***conclusions*** and viceversa!

- This is a consequence of the fact that in classical logic ***negation is an involution***.

- Thanks to ***DeMorgan's Laws***:

$$\neg(A \wedge B) = \neg A \vee \neg B \qquad \neg(A \vee B) = \neg A \wedge \neg B$$

  negations can be pushed ***inside*** a formula, taking the principal connective ***on top***.

- ***Simplification***: negations appear only on ***atomic*** formulas.

The right-side calculus

Applying the following rule until the left side is ***empty***:

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}$$

We get the ***right-sided calculus***:

- ***Identity group***:

$$\frac{}{\vdash A, \neg A}(Ax) \qquad \frac{\vdash A, \Gamma \quad \vdash \neg A, \Delta}{\vdash \Gamma, \Delta}(cut)$$

- ***Logical Group***:

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \wedge B, \Gamma, \Delta}(\wedge) \qquad \frac{\vdash A, B, \Gamma}{\vdash A \vee B, \Gamma}(\vee)$$

- ***Structural Group and Truth Values***:

$$\frac{\vdash A, A, \Gamma}{\vdash A, \Gamma}(Contr) \qquad \frac{\vdash \Gamma}{\vdash A, \Gamma}(Weak) \qquad \frac{}{\vdash \top} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}$$

Additive presentation

The logical rules

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \wedge B, \Gamma, \Delta}(\wedge) \qquad\qquad \frac{\vdash A, B, \Gamma}{\vdash A \vee B, \Gamma}(\vee)$$

admit an ***alternative presentation***:

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Gamma}{\vdash A \wedge B, \Gamma}(\wedge) \qquad\qquad \frac{\vdash A, \Gamma}{\vdash A \vee B, \Gamma}(\vee 1) \qquad\qquad \frac{\vdash B, \Gamma}{\vdash A \vee B, \Gamma}(\vee 2)$$

Key point: the management of the ***context***.

The first presentation is called ***multiplicative***, the second ***additive***.

The ***structural rules*** prove that the two presentations are ***equivalent***.

Simulation of rules 1

Additive + weakening $\Rightarrow$ ***Multiplicative***:

- Take the ***multiplicative premises***:

$$\vdash A, \Gamma \qquad \vdash B, \Delta$$

- Repeated applications of ***weakening*** get:

$$\vdash A, \Gamma, \Delta \qquad \vdash B, \Gamma, \Delta$$

- The ***additive*** $\wedge$ rule gets:

$$\vdash A \wedge B, \Gamma, \Delta$$

Which is the ***multiplicative conclusion***.

Simulation of rules 2

Multiplicative + contraction $\Rightarrow$ ***Additive***:

- Take the ***additive premises***:

$$\vdash A, \Gamma \qquad \vdash B, \Gamma$$

- Apply the ***multiplicative*** $\wedge$ rule:

$$\vdash A \wedge B, \Gamma, \Gamma$$

- Repeated applications of ***contraction*** get:

$$\vdash A \wedge B, \Gamma$$

Which is the ***additive*** conclusion.

What if?

- ***Structural rules*** $\Rightarrow$ ***additive = multiplicative***.
- What if we ***eliminate*** the structural rules?
- The two sides of the "$\vdash$" become ***multisets***.
- We get two ***non equivalent*** presentations of... of what?
- ***Not classical logic***, because weakening and contraction are fundamental rules
- Two new systems:

  Multiplicative Linear Logic and ***Additive Linear Logic***

Outline

Multiplicative LL

Identity group:

$$\frac{}{\vdash A, A^{\perp}}(Ax) \qquad \frac{\vdash A, \Gamma \quad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}(cut)$$

Logical Group:

$$\frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}(\otimes) \qquad \frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}(\parr)$$

Remark: commas are pars.

Units:

$$\frac{}{\vdash 1} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}$$

Additive LL

Identity group:

$$\frac{}{\vdash A, A^{\perp}}(Ax) \qquad \frac{\vdash A, \Gamma \quad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}(cut)$$

Logical Group:

$$\frac{\vdash A, \Gamma \quad \vdash B, \Gamma}{\vdash A \& B, \Gamma}(\&) \qquad \frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma}(\oplus 1) \qquad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma}(\oplus 2)$$

Units:

$$\frac{}{\vdash \Gamma, \top}$$

There is ***another unit***, 0, but there is ***no rule*** for it.

Remark: Additive LL is somehow ***degenerated***, any sequent has exactly ***two*** formulas!

MALL

- The two systems ***share*** the axiom and the cut rule.
- They can put together, obtaining Multiplicative Additive Linear Logic, ***MALL*** in short:
  - Linear negation, noted $()^{\perp}$, is involutive.
  - DeMorgan's laws are avaible:

$$(A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp} \qquad (A \& B)^{\perp} = A^{\perp} \oplus B^{\perp}$$

$$(A \parr B)^{\perp} = A^{\perp} \otimes B^{\perp} \qquad (A \oplus B)^{\perp} = A^{\perp} \& B^{\perp}$$

  - There are ***four*** units, 1, $\perp$, $\top$ and 0.
  - Each one is the ***neutral element*** of a connective.
  - Example: $A \otimes 1$ is provable ***iff*** A is provable.

Intuitions about the multiplicatives

- ***$A \otimes B$ means***:

  *You have **exactly** one copy of A and one of B, **no more, no less***

- ***None*** of $(A \otimes B) \multimap A$, $(A \otimes A) \multimap A$ and $A \multimap (A \otimes A)$ is provable.

- The tensor is ***commutative***, ***associative*** and it has 1 as neutral element, so over the set of formulas it realizes the ***free commutative monoid***, *i.e.* it defines ***multisets of formulas***.

- $A \parr B$ ***has no intuitive meaning***.

- It defines ***linear implication*** $A \multimap B := A^{\perp} \parr B$.

- ***$A \multimap B$ means***:

  *Having **exactly** one copy of A, it can be **used**, and **consumed**, to produce **exactly** one copy of B*

Intuitions about the additives

- ***$A \& B$ means***:

  *You have **one among** A and B, and **you can choose** which one, but **you cannot have both***

- $(A\&B) \multimap A$, $(A\&A) \multimap A$ and $A \multimap (A\&A)$ are provable. But $A \multimap (A\&B)$ is not provable, nor is $(A \otimes B) \multimap (A\&B)$: of course we can have one among A and B, but we ***cannot discard the other***.
- $A\&B$ is ***not*** a disjunction: $(A\&B) \multimap A$ and $(A\&B) \multimap B$ are both ***provable***.
- ***$A \oplus B$ means***:

  *You have **exactly one** among A and B, but **you don't know which one***

  this is the disjunction

- $(A\&B) \multimap (A \oplus B)$, $(A \oplus A) \multimap A$ and $A \multimap (A \oplus B)$ are provable, but $(A \oplus B) \multimap (A\&B)$, $(A \oplus B) \multimap A$ are ***not***.

Linear implication

- There is ***only one implication***, defined by the multiplicative disjunction as $A \multimap B := A^{\perp} ⅋ B$.
- Why isn't there an additive implication? Every implication whatsoever, noted now $\Rrightarrow$, has to satisfy at least $A \Rrightarrow A$. The ***additive implication*** would be $A \Rrightarrow B := A^{\perp} \oplus B$ but $A^{\perp} \oplus A$ is ***not*** provable.
- Given $A \multimap B$ and $A \multimap C$, one can infer $A \otimes A \multimap B \otimes C$, but not $A \multimap B \otimes C$.
- That is:

*If **paying 5 euros I can have cigarettes**, and*
***paying 5 euros I can go to the cinema**,*
it is right that only
paying 10 euros I can have cigarettes** and **go to the cinema

States: the chemistry example

- Take ***basic chemistry***, with ***reactions*** like $2H_2 + O_2 \rightarrow 2H_2O$.
- The coding $H_2 \wedge H_2 \wedge O_2 \Rightarrow H_2O \wedge H_2O$ ***does not work*** because $\wedge$ is ***idempotent***, *i.e.* $H_2 \wedge H_2 \Leftrightarrow H_2$.
- Moreover, the implication ***does not*** consume the hypothesis, *i.e.* one gets $2H_2 \wedge O_2 \wedge 2H_2O$.
- Classical logic ***cannot*** represent the ***updating of the state***.
- Instead $H_2 \otimes H_2 \otimes O_2 \multimap H_2O \otimes H_2O$ ***works perfectly***!

Outline

Beyond MALL

- MALL is a ***nice system***, but it ***cannot*** represent classical reasoning.
- Furthermore, the multiplicative and the additive are in some sense ***apart***: there is ***no way*** to relate a multiplicative connective with an additive one.
- ***Solution***: to re-introduce ***weakening*** and ***contraction*** but only on some ***marked*** formulas.
- The ‘markers’ are two dual ***modalities***, $!$ and $?$, allowing a formula to be used ***any number of times***. The rules:

$$\frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}(der) \qquad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}(prom)$$

$$\frac{\vdash \Gamma}{\vdash ?A, \Gamma}(weak) \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}(contr)$$

Intuitions on exponentials

- $!$ and $?$ are called ***exponential connectives***.
- ***Linear Logic*** is MALL plus the exponentials.
- Intuitions:
  - $?$ marks ***hypothesis*** which are relieved from their linear status, *i.e.* they can be ***copied*** and ***discarded*** at will.
  - While $!$ marks ***conclusions*** which can be obtained ***as many times as you want***.
- Those formulas are ***provable***:

$$!A \multimap A \otimes A \qquad !A \multimap (!A \otimes !A) \qquad !A \multimap !(A \otimes A)$$

What exponentials add to the picture

- The exponentials ***relates*** the multiplicative and the additive logics.
- Indeed, the following formula is provable:

$$!(A \& B) = !A \otimes !B$$

  It is called ***the fundamental isomorphism***.
- It is the reason for the ***denominations*** of the connectives.
- Compare with:

$$e^{A+B} = e^{A} \cdot e^{B}$$

Outline

Complexities of CL

A ***measure of the expressiveness*** of a logical system is the ***complexity*** of the provability problem.

Results about Classical Logic:

- ***Constant-only*** Classical Logic is ***linear***.
- ***Propositional*** Classical Logic is ***NP-complete***.
- ***First-order*** Classical Logic is ***undecidable***.

First-order classical logic is often too ***expressive***.

While the propositional one is often too ***weak***.

Complexities of LL

Linear logic presents a more ***colourful panorama***:

- MLL, and FO-MLL, are ***NP-complete***.
- MALL is PSPACE-complete, and FO-MALL is ***NEXPTIME-complete***.
- MELL is ***EXPSPACE-hard***, but the upper-bound is ***unknown***.
- Propositional LL is ***undecidable*** (and obviously so is the FO version).
- ***Constant-only*** variations ***do not decrease*** the complexity for any fragment.

Outline

Cuts and how to avoid them 1

- The cut rule

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

  is a generalization of the modus ponens.

- Sequent calculus' fundamental property:

  Any provable formula has a proof ***without the cut rule***

- This means:

  Transitivity is not needed anymore

- Moreover: there is an ***algorithm*** taking a proof ***with cuts*** and producing a proof ***without cuts***.
- This result is ***very strange***: in Hilbert systems modus ponens is the ***only tool*** for reasoning!

Cuts and how to avoid them 2

- The cut rule is the ***only*** rule where the premises contain a formula ***not*** in the conclusion.
- The other rules use only ***subformulas*** of the conclusion.
- Cut-elimination gives a way to ***find proofs*** of a formula A in an ***automated way***.
- If a proof exists, then there is one proof ***without cuts*** where all the involved formulas are subformulas of A.
- This is called the ***Subformula Property***.

Outline

The identity of proofs

Consider these two proofs:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^{\perp}}\,\text{ax} \qquad \dfrac{}{\vdash B, B^{\perp}}\,\text{ax}}{\vdash A^{\perp}, B^{\perp}, A \otimes B}\,\otimes}{\vdash A^{\perp} \parr B^{\perp}, A \otimes B}\,\parr \qquad \dfrac{}{\vdash C, C^{\perp}}\,\text{ax}}{\vdash A^{\perp} \parr B^{\perp}, C^{\perp}, (A \otimes B) \otimes C}\,\otimes$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^{\perp}}\,\text{ax} \qquad \dfrac{}{\vdash B, B^{\perp}}\,\text{ax}}{\vdash A^{\perp}, B^{\perp}, A \otimes B}\,\otimes \qquad \dfrac{}{\vdash C, C^{\perp}}\,\text{ax}}{\vdash A^{\perp}, B^{\perp}, C^{\perp}, (A \otimes B) \otimes C}\,\otimes}{\vdash A^{\perp} \parr B^{\perp}, C^{\perp}, (A \otimes B) \otimes C}\,\parr$$

Are they ***the same***?

Sequent calculus and cut-elimination

- Cut-elimination in sequent calculus is ***heavy***.
- There are two cases of cut-elimination, ***key*** and ***commutative***.
- A ***key*** case:

$$
\dfrac{
\otimes\,\dfrac{\begin{matrix}\vdots\,\pi_1\\ \vdash \Gamma_1, A\end{matrix} \qquad \begin{matrix}\vdots\,\pi_2\\ \vdash \Gamma_2, B\end{matrix}}{\vdash \Gamma_1, \Gamma_2, A \otimes B}
\qquad
\dfrac{\begin{matrix}\vdots\,\theta\\ \vdash \Delta, A^{\perp}, B^{\perp}\end{matrix}}{\vdash \Delta, A^{\perp} \parr B^{\perp}}\,\parr
}{\vdash, \Gamma_1, \Gamma_2, \Delta}\,\text{cut}
$$

Whose ***elimination*** is:

$$
\dfrac{
\begin{matrix}\vdots\,\pi_2\\ \vdash \Gamma_2, B\end{matrix}
\qquad
\dfrac{\begin{matrix}\vdots\,\pi_1\\ \vdash \Gamma_1, A\end{matrix} \qquad \begin{matrix}\vdots\,\theta\\ \vdash \Delta, A^{\perp}, B^{\perp}\end{matrix}}{\vdash, \Gamma_1, \Delta, B^{\perp}}\,\text{cut}
}{\vdash \Gamma_1, \Gamma_2, \Delta}\,\text{cut}
$$

Sequent calculus and cut-elimination

A ***commutative*** case:

$$\dfrac{\dfrac{\overset{\vdots}{\underset{\vdots}{\pi}}}{\vdash \Gamma, C, D, A \otimes B}}{\vdash \Gamma, C \parr D, A \otimes B}\parr \quad \dfrac{\dfrac{\overset{\vdots}{\underset{\vdots}{\theta}}}{\vdash \Delta, A^{\perp}, B^{\perp}}}{\vdash \Delta, A^{\perp} \parr B^{\perp}}\parr$$
$$\overline{\vdash, \Gamma, \Delta, C \parr D}\ \text{cut}$$

Remark: in the left proof the last rule is not the one introducing the tensor. This cut reduces to:

$$\dfrac{\dfrac{\overset{\vdots}{\underset{\vdots}{\pi}}}{\vdash \Gamma, C, D, A \otimes B} \quad \dfrac{\dfrac{\overset{\vdots}{\underset{\vdots}{\theta}}}{\vdash \Delta, A^{\perp}, B^{\perp}}}{\vdash \Delta, A^{\perp} \parr B^{\perp}}\parr}{\dfrac{\vdash, \Gamma, \Delta, C, D}{\vdash \Gamma, C \parr D}\parr}\text{cut}$$

The identity of proofs

- Sequent calculus proofs are ***too sequential***.
- Sequentiality introduces differences which are ***not relevant***.
- It also induces ***commutative*** cut-elimination cases.
- These cases introduce ***many complications*** in the study of cut-elimination.
- Can we represent proofs in a ***better*** way?
- ***Idea***: represent only the ***causality*** relation between rules.
- This requires to switch to a ***graphical representation***.

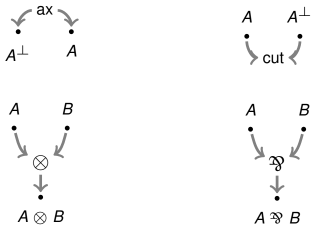
$\mathrm{MLL}^{\neg\{1,\bot\}}$

- $\mathrm{MLL}^{\neg\{1,\bot\}}$ ***rules***:

$$\frac{}{\vdash A^\bot, A}\ \mathrm{ax} \qquad\qquad \frac{\vdash \Gamma, A \qquad \vdash A^\bot, \Delta}{\vdash \Gamma, \Delta}\ \mathrm{cut}$$

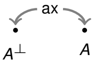
$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes \qquad\qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr$$

- ***Nets*** for $\mathrm{MLL}^{\neg\{1,\bot\}}$ are built out of ***links***:

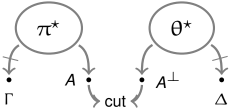


The translation

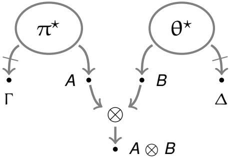
$$\left(\frac{}{\vdash A^{\perp}, A}\,\mathsf{ax}\right)^{\star} =$$



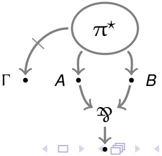
$$\left(\dfrac{\begin{array}{c}\vdots\ \pi\\ \vdash \Gamma, A\end{array} \qquad \begin{array}{c}\vdots\ \theta\\ \vdash A^{\perp}, \Delta\end{array}}{\vdash \Gamma, \Delta}\,\mathsf{cut}\right)^{\star} =$$



$$\left(\dfrac{\begin{array}{c}\vdots\ \pi\\ \vdash \Gamma, A\end{array} \qquad \begin{array}{c}\vdots\ \theta\\ \vdash \Delta, B\end{array}}{\vdash \Gamma, \Delta, A \otimes B}\,\otimes\right)^{\star} =$$



$$\left(\dfrac{\begin{array}{c}\vdots\ \pi\\ \vdash \Gamma, A, B\end{array}}{\vdash \Gamma, A \parr B}\,\parr\right)^{\star} =$$



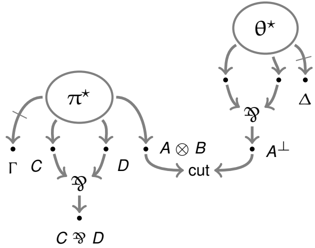
Proof-nets and commutative cut-elimination

The commutative reduction:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \pi \\ \vdots\end{matrix}}{\vdash \Gamma, C, D, A \otimes B}\;}{\vdash \Gamma, C \parr D, A \otimes B}\parr \qquad \dfrac{\dfrac{\begin{matrix}\vdots\\ \theta \\ \vdots\end{matrix}}{\vdash \Delta, A^{\perp}, B^{\perp}}}{\vdash \Delta, A^{\perp} \parr B^{\perp}}\parr$$

$$\dfrac{\vdash \Gamma, C \parr D, A \otimes B \qquad \vdash \Delta, A^{\perp} \parr B^{\perp}}{\vdash, \Gamma, \Delta, C \parr D}\text{cut} \quad \rightarrow \quad \dfrac{\dfrac{\begin{matrix}\vdots\\ \pi \\ \vdots\end{matrix}\;\;\vdash \Gamma, C, D, A \otimes B \qquad \dfrac{\begin{matrix}\vdots\\ \theta \\ \vdots\end{matrix}\;\;\vdash \Delta, A^{\perp}, B^{\perp}}{\vdash \Delta, A^{\perp} \parr B^{\perp}}\parr}{\vdash, \Gamma, \Delta, C, D}\text{cut}}{\vdash \Gamma, C \parr D}\parr$$

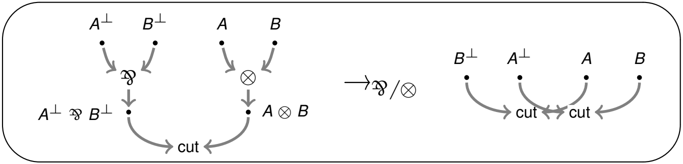
Both proofs translate to ***the same net***:

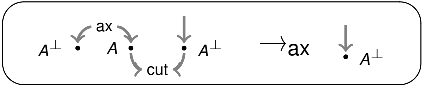


And so commutative cases simply ***vanish***!

Proof-nets and key cut-elimination

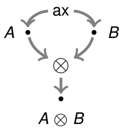
- The ***key*** cut-elimination cases do ***not*** vanish.
- They can be seen as cut-elimination ***on proof-nets***.
- Key cut-elimination rules:





Correctness criterions

- There are ***more*** $\mathrm{MLL}^{\neg\{1,\perp\}}$ ***nets than*** $\mathrm{MLL}^{\neg\{1,\perp\}}$ ***proofs***.
- For instance:



- It is interesting to ***characterize*** the graphs corresponding to proofs in ***non-inductive ways***.
- ***Correctness Criterion*** = set of ***geometrical*** conditions characterizing the graphs corresponding to proofs.
- Curiously, correctness is a ***global*** property.
- Correctness for MLL is related to ***acyclicity*** and ***connectedness***.

Correctness criterions

- MLL admits ***many*** correctness criterions.
- A characterization is proved to be a criterion by defining a ***sequentialization*** procedure.
- ***Sequentialization***: an algorithm which ***extracts*** from a correct net G a proof π_G which ***translates*** to G.
- ***Weakness*** of LL: essentially ***only MLL*** admits correctness criterions.
- ***Intuitionistic*** Linear Logic has a much more ***well-behaved*** geometrical theory.