

# On automating the extraction of programs from termination proofs\*

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## Abstract

We investigate an automated program synthesis system that is based on the paradigm of programming by proofs. To automatically extract a  $\lambda$ -term that computes a recursive function given by a set of equations the system must find a formal proof of the totality of the given function. Because of the particular logical framework, usually such approaches make it difficult to use termination techniques such as those in rewriting theory. We overcome this difficulty for the automated system that we consider by exploiting product types. As a consequence, this would enable the incorporation of termination techniques used in other areas while still extracting programs.

**Keywords:** *Program extraction, product types, termination, ProPre system.*

## 1 Introduction

The Curry-Howard isomorphism [3] that establishes a correspondence between programs and proofs of specifications plays a major role in many type systems. Programming methods using the *proof as program* paradigm ensure some correctness of programs extracted from a proof of function totality and provide a logical framework for which the behaviour of programs can be analysed. Of these systems which exploit the proof as program paradigm, we mention *Second Order Functional Arithmetic (AF2)*, cf. [7, 9] and a faithful extension of *AF2* called *Recursive Type Theory (TTR)*, cf. [16]. Both systems use equations as algorithmic specifications. In *AF2* and *TTR*, the compilation phase corresponds to formal termination proofs of the specifications of functions from which  $\lambda$ -terms that compute the functions are extracted.

Using the logical framework of *TTR*, an automated system called *ProPre*, has been developed by P. Manoury and M. Simonot [12, 11]. The automated termination problem turns out to be a major issue in the development of the system. Alongside the system, where data types and specifications of functions are introduced by the user in an ML-style, an algorithm has been designed using strategies to search for formal termination proofs for each specification. When the system succeeds in developing a formal termination proof for a specification, a  $\lambda$ -term that computes the function is given.

As mentioned in [12], the automated termination proofs in this system differ from the usual techniques of rewriting systems because they have to follow several requirements. They must be *proofs of totality* in order to enable the extraction of  $\lambda$ -terms. In *ProPre*, one has to make sure not only that the programs will give an output for any input, but also that

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for any well-typed input the result will also be well-typed. Finally, the proofs must also be expressed in a formal logical framework, namely, the natural deduction style. The  $\lambda$ -terms are obtained from the proof trees that are built in a natural deduction style according to the recursive type theory *TTR*.

Therefore enhancing automated proofs strategy is a central issue in programming languages like *AF2* or *TTR*. While termination methods for functional programming based on ordinal measures have been developed in [14, 5] relating to the formal proofs devised in [12, 13], the purpose of this paper is to analyse in some sense the reverse of the question. That is, we analyse the possibility to incorporate new termination techniques for the extraction of programs in the *ProPre* or *TTR* context.

In order to simplify the analysis of the formal proofs obtained in the logical framework of *ProPre*, we show that the kernel of these formal proofs, called *formal terminal state property (ftsp)*, can be abstracted using a simple data structure. This gives rise to a simple termination property, which we call *abstract terminal state property (atsp)*. The interest of *atsp* is that on one hand the termination condition is sufficient to show the termination using the ordinal measures of [5] independently of the particular logical framework of *ProPre*, and on the other hand we also prove that we can automatically reconstruct a formal proof directly from an *atsp* so that a lambda term can be extracted. That is to say the first result of this paper is to establish a correspondence between *atsp* with a class of ordinal measures in a simple context for the termination and the formal proofs built in *ProPre*.

This correspondence implies that the termination proofs of recursive functions obtained in [4] do not admit in general a formal proof in *ProPre*. Indeed the class of these functions is larger than those proved with the class of ordinal measures of [14, 5]. To overcome the fact that there is in general no formal proof in *ProPre* for these functions, the second result presented in this paper allows the synthesis of these functions still making use of the whole framework of *ProPre* but in a different way in *TTR*. Actually the result turns out to be stronger since it can be applied for recursive functions whose termination is proved by other automated methods such as techniques coming from rewriting theory (see e.g. [1]). The principle consists in simulating a semantic method. That is, from a well-founded ordering for which each recursive call is decreasing, one must be able to build a formal proof by considering general induction on tuples of arguments of the function. Though the principle is natural, this approach becomes difficult when we want in particular to extract programs because we have to take into account the logical framework and the structures of the proofs that we may or may not be able to build.

## 2 Logical framework

We briefly present the *ProPre* system (see [10, 12, 13] for details). *ProPre* relies on the *proofs as programs* paradigm that exploits the Curry-Howard isomorphism and deals with the recursive type theory *TTR* [16]. In *ProPre*, the user needs to only define data types and functions.  $\lambda$ -terms are automatically extracted from the formal proofs of the termination statements of functions which can be viewed as the compilation part.

*ProPre* deals with recursive functions. The data types and functions are defined in an ML like syntax. For instance, if *N* denotes the type of natural numbers, then the list of natural numbers is defined by:

$$\text{Type Ln} : \text{Nil} \mid \text{Cons N Ln};$$

and the *append* function is specified by:

$$\begin{aligned} \text{Let } \text{append} & : \text{Ln, Ln} \rightarrow \text{Ln} \\ \text{Nil } y \Rightarrow y & \mid (\text{Cons } n \ x) \ y \Rightarrow (\text{Cons } n \ (\text{append } x \ y)); \end{aligned}$$

Once a data type is introduced by the user, a second order formula is automatically generated. E.g., the following second order formula is automatically generated and associated to the list of natural numbers:

$$Ln(x) := \forall X(X(nil) \rightarrow (\forall n(N(n) \rightarrow \forall y(X(y) \rightarrow X(cons(n, y)))))) \rightarrow X(x).$$

This formula stands for the least set that contains the *nil* element and is closed under the constructor *cons*. Each data type will be abbreviated by a unary data symbol, as it is for instance with the symbol *N* that represents the data type of natural numbers. Furthermore, once a function is specified in the system, a *termination statement* is automatically produced [11]. As an example, the termination statement of the *append* function is the formula:

$$\forall x(Ln(x) \rightarrow \forall y(Ln(y) \rightarrow Ln(append(x, y)))).$$

The system then attempts to prove the termination statement of the function using the set of equations that define the function. In a successful case, a  $\lambda$ -term that computes the function is synthesized from the building of a formal proof in a natural deduction style [12]. Informally, if *T* is a  $\lambda$ -term obtained for the function *append* and *t*<sub>1</sub>, *t*<sub>2</sub> are  $\lambda$ -terms that respectively model terms *u*<sub>1</sub> of type **Ln** and *u*<sub>2</sub> of type **Ln**, then the  $\lambda$ -term ((*T t*<sub>1</sub>) *t*<sub>2</sub>) reduces to a normal form *V* that represents the value of *append*(*u*<sub>1</sub>, *u*<sub>2</sub>) of type **Ln**.

We refer the reader to [7, 8, 15, 16] regarding the theory that allows to derive  $\lambda$ -terms from termination proofs of the specification in a natural deduction style.

## 2.1 The typing rules of *AF2*

We recall the typing rules of *AF2* which are also part of *TTR*. We assume a set  $\mathcal{F}$  of function symbols and a countable set  $\mathcal{X}$  of individual variables. The logical terms are inductively defined as follows:

- individual variables are logical terms;
- if *f* is an *n*-ary function symbol in  $\mathcal{F}$  and *t*<sub>1</sub>, ..., *t*<sub>*n*</sub> are logical terms, then *f*(*t*<sub>1</sub>, ..., *t*<sub>*n*</sub>) is a logical term.

We also assume a countable set of predicate variables. Formulas are inductively defined as follows:

- if *X* is an *n*-ary predicate variable and *t*<sub>1</sub>, ..., *t*<sub>*n*</sub> are logical terms, then *X*(*t*<sub>1</sub>, ..., *t*<sub>*n*</sub>) is a formula;
- if *A* and *B* are formulas then *A* → *B* is a formula;
- if *A* is a formula and  $\eta$  is a first or second order variable, then  $\forall \eta A$  is a formula.

We will use  $\forall x A \rightarrow B$  to denote  $\forall x(A \rightarrow B)$ . For convenience, a formula of the form  $F_1 \rightarrow (F_2 \rightarrow \dots (F_{n-1} \rightarrow F_n) \dots)$  will also be denoted by  $F_1, \dots, F_n \rightarrow F$ . For instance  $\forall x D_1(x), \forall y D_2(y) \rightarrow F$  stands for the formula  $\forall x(D_1(x) \rightarrow \forall y(D_2(y) \rightarrow F))$ .

A typing judgment is an expression of the form: “ $x_1 : F_1, \dots, x_n : F_n \vdash_{\mathcal{E}} t : F$ ”, where *x*<sub>1</sub>, ..., *x*<sub>*n*</sub> are distinct  $\lambda$ -variables, *t* is a  $\lambda$ -term, *F*, *F*<sub>1</sub>, ..., *F*<sub>*n*</sub> are formulas and  $\mathcal{E}$  is a set of equations on logical terms. The left-hand side of the judgment is called the context. Note that we can freely use the same notation for both the  $\lambda$ -terms and the logical terms which occur in the formulas, as the context will clarify whether a term is a  $\lambda$ -term or a logical term. In particular, the word “variable” may also refer to a “ $\lambda$ -variable”. The typing rules of *AF2* are given in Table 1 where  $\mathcal{E}$  is a set of equations on logical terms.

$\frac{}{\Gamma, x : A \vdash_{\mathcal{E}} x : A} \quad (ax)$	$\frac{\Gamma \vdash_{\mathcal{E}} t : A[u/y] \quad \mathcal{E} \vdash u = v}{\Gamma \vdash_{\mathcal{E}} t : A[v/y]} \quad (eq)$
$\frac{\Gamma, x : A \vdash_{\mathcal{E}} t : B}{\Gamma \vdash_{\mathcal{E}} \lambda x.t : A \rightarrow B} \quad (\rightarrow_i)$	$\frac{\Gamma \vdash_{\mathcal{E}} u : A \quad \Gamma \vdash_{\mathcal{E}} t : A \rightarrow B}{\Gamma \vdash_{\mathcal{E}} (t u) : B} \quad (\rightarrow_e)$
$\frac{\Gamma \vdash_{\mathcal{E}} t : A}{\Gamma \vdash_{\mathcal{E}} t : \forall y A} \quad (\forall_i^1)$	$\frac{\Gamma \vdash_{\mathcal{E}} t : \forall y A}{\Gamma \vdash_{\mathcal{E}} t : A[\tau/y]} \quad (\forall_e^1)$
$\frac{\Gamma \vdash_{\mathcal{E}} t : A}{\Gamma \vdash_{\mathcal{E}} t : \forall Y A} \quad (\forall_i^2)$	$\frac{\Gamma \vdash_{\mathcal{E}} t : \forall Y A}{\Gamma \vdash_{\mathcal{E}} t : A[T/Y]} \quad (\forall_e^2)$

Table 1: Rules of the Second Order Functional Arithmetic (AF2)

$\frac{\Gamma \vdash_{\mathcal{E}} t : A \quad \Gamma \vdash_{\mathcal{E}} e}{\Gamma \vdash_{\mathcal{E}} t : A \upharpoonright e} \quad (\upharpoonright_1)$	$\frac{\Gamma \vdash_{\mathcal{E}} t : A \upharpoonright e}{\Gamma \vdash_{\mathcal{E}} t : A} \quad (\upharpoonright_2)$	$\frac{\Gamma \vdash_{\mathcal{E}} t : A \upharpoonright e}{\Gamma \vdash_{\mathcal{E}} e} \quad (\upharpoonright_3)$
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Table 2: Rules of the hiding operator  $\upharpoonright$ .

In Table 1,  $\Gamma$  is a context of the form  $x_1 : A_1, \dots, x_n : A_n$  and may be empty;  $y$  (resp.  $Y$ ) is a first (resp. second) order variable not occurring free in  $A_1, \dots, A_n$ ;  $\tau, u, v$  are first order terms and  $T$  is a formula. The expression  $\mathcal{E} \vdash u = v$  means that the equation  $u = v$  is derivable from  $\mathcal{E}$  in second order logic. For more explanations we refer to [7].

The types and formal data types play an important role in *AF2* and *TTR* in relation to a notion of *realizability* [8] that ensures the extracted  $\lambda$ -terms compute the defined functions. However, for the sake of clarity, we do not state here the definition of formal data types and the realizability notion whose details can be found in [7, 8]. Now, for an  $n$ -ary function  $f$  if we have the typing judgment:

$$\vdash_{\mathcal{E}} t : \forall x_1 \dots \forall x_n (D_1(x_1) \rightarrow (\dots \rightarrow (D_n(x_n) \rightarrow D(f(x_1, \dots, x_n)))) \dots)$$

for some  $\lambda$ -term  $t$  where  $D_1, \dots, D_n, D$  denote formal data types, then the  $\lambda$ -term  $t$  computes the function  $f$  according to the set  $\mathcal{E}$ .

## 2.2 Some rules of *TTR*

As for *AF2*, we do not state the data types and the realizability notion of *TTR*. In particular we do not give the second order least fixed point operator  $\mu$  (see [15]) which allows one to define the data types which are represented here by unary data symbols  $D, D', \dots, D_1, \dots, D_n$ . Furthermore, for the sake of presentation we do not state all the rules (which also include those of *AF2*), but only give those needed for our purpose.

In *TTR*, a binary symbol  $\upharpoonright$ , called *hiding operator* in [15], is added. Its meaning is a conjunction which only keeps the algorithmic contents of the left part in order to prevent unnecessary algorithmic content of the termination proof to be carried out in the  $\lambda$ -terms (see [16, 12]). The rules related to the hiding operator are given in Table 2.

The hiding operator is used with a relation  $\prec$  where the definition of formulas given in section 2.1 is now completed as follows:

If  $A$  is a formula, and  $u, v$  are terms then  $A \uparrow (u \prec v)$  is a formula.

If  $A$  is a formula where a distinguished variable  $x$  occurs, we abbreviate the formula  $A[u/x] \uparrow (u \prec v)$  with the notation  $Au_{\prec v}$ .

Among the rules of *TTR*, several rules are used to reproduce, from the programming point of view, the reasoning by induction. The rule below stands in *TTR* for an external induction rule where the relation  $\prec$  denotes a well-founded partial ordering on the terms of the algebra:

$$\frac{\Gamma \vdash_{\mathcal{E}} t : \forall x[\forall z[Dz_{\prec x} \rightarrow B[z/x]] \rightarrow [D(x) \rightarrow B]]}{\Gamma \vdash_{\mathcal{E}} (T t) : \forall x[D(x) \rightarrow B]} \quad (Ext)$$

In the rule (*Ext*), the lambda term  $T$  is the Turing fixed-point operator,  $D$  is a data type and  $x$  is a variable not occurring in the formula  $B$ .

From the (*Ext*) rule, it is possible to derive the  $\widetilde{Ind}$  formula:  
 $\widetilde{Ind} := \forall x(D^r(x) \rightarrow \forall X(\forall y(D^r(y) \rightarrow \forall z(D^r z_{\prec y} \rightarrow X(z)) \rightarrow X(y)) \rightarrow X(x))$ .

That is, for each recursive data type, there is a  $\lambda$ -term  $ind$  such that:  $\vdash_{\mathcal{E}} ind : \widetilde{Ind}$  for any set  $\mathcal{E}$  of equations. We say that the term  $ind$  witnesses the proof of  $\widetilde{Ind}$ . This is stated with Lemma 2.1 below, which is presented with the type of natural numbers in [15].

**Lemma 2.1.** For each recursive data type, there exists a  $\lambda$ -term  $ind$  such that:  
 $\vdash_{\mathcal{E}} ind : \widetilde{Ind}$  for any set  $\mathcal{E}$  of equations.

Lemma 2.1 can be proven for  $\mathcal{E}$  being empty. Then, one can use the result of [7] which states that if  $\mathcal{E}_1 \subseteq \mathcal{E}_2$  then  $\vdash_{\mathcal{E}_1} t : P$  implies  $\vdash_{\mathcal{E}_2} t : P$ . The proof of Lemma 2.1 in [15], given only with the type of natural numbers, can actually also be applied to any data type. In particular if  $T$  is the Turing fixed-point operator, then the lambda term  $ind = (T \lambda x \lambda y \lambda z ((z y) \lambda m ((x m) z)))$ , is valid for any data type  $D$ . The above Lemma is useful for the definition of a macro-rule, called the *Ind*-rule, in the *ProPre* system.

## 2.3 The *ProPre* system

We assume that the set of functions  $\mathcal{F}$  is divided into two disjoint sets, the set  $\mathcal{F}_c$  of *constructor symbols* and the set  $\mathcal{F}_d$  of *defined function symbols* also called defined functions. Each function  $f$  is supposed to have a type denoted by  $D_1, \dots, D_n \rightarrow D$  where  $D_1, \dots, D_n, D$  denote data symbols and  $n$  denotes the arity of the function  $f$ . We may write  $f : D_1, \dots, D_n \rightarrow D$  to both introduce a function  $f$  and its type  $D_1, \dots, D_n \rightarrow D$ .

**Definitions 2.2.** [**Specification; termination statement; recursive call**]

- A *specification*  $\mathcal{E}_f$  of a defined function  $f : D_1, \dots, D_n \rightarrow D$  in  $\mathcal{F}_d$  is a non overlapping set of left-linear equations  $\{(e_1, e'_1), \dots, (e_p, e'_p)\}$  such that for all  $1 \leq i \leq p$ ,  $e_i$  is of the form  $f(t_1, \dots, t_n)$  where  $t_j$  is a constructor term (i.e. without occurrences of defined function symbols) of type  $D_j$ ,  $j = 1, \dots, n$ , and  $e'_i$  is a term of type  $D$ .
- The *termination statement* of a function  $f : D_1, \dots, D_n \rightarrow D$  is the formula  $\forall x_1(D_1(x_1) \rightarrow \dots \rightarrow \forall x_n(D_n(x_n) \rightarrow D(f(x_1, \dots, x_n))))$ .
- Let  $\mathcal{E}_f$  a specification of a function  $f$ . A *recursive call* of  $f$  is a pair  $(t, v)$  where  $t$  is the left-hand side of an equation  $(t, u)$  of  $\mathcal{E}_f$  and  $v$  a subterm of  $u$  of the form  $f(v_1, \dots, v_n)$ .

An equation  $(l, r)$  of a specification may be written  $l = r$  (as an equational axiom in *TTR*). We may also drop the brackets to ease the readability.

The formal proofs of *ProPre*, called I-proofs, are built upon distributing trees, based on two main rules derived from the *TTR Struct* rule and the *Ind* rule in [12]. The distributing trees built in *ProPre* are characterized by a property called *formal terminal state property*. This section presents these two main rules, the distributing trees and the formal terminal state property. Let us first introduce some notations.

**Notation 2.3.** If  $P$  is the formula  $F_1, \dots, F_k, \forall x D'(x), F_{k+1}, \dots, F_m \rightarrow D(t)$ , then  $P_{-D'(x)}$ , will denote the formula  $F_1, \dots, F_k, F_{k+1}, \dots, F_m \rightarrow D(t)$ .

The above notation is correct as it will be used at the same time when the quantified variable  $x$  will be substituted by a term in the formula  $P_{-D(x)}$  with respect to the context (cf. next two lemmas with Notation 2.4) or when the variable  $x$  will be introduced in the context.

**Notation 2.4.** Let  $C$  be a constructor symbol of a type  $D_1, \dots, D_k \rightarrow D$ . Let  $x_1, \dots, x_k, z$  be distinct variables. Let  $F(x)$  be a formula in which the variable  $x$  is free and the variables  $z, x_1, \dots, x_k$  do not occur and let  $t = C(x_1, \dots, x_k)$ . Then  $\Phi_C(F(x))$  and  $\Psi_C(F(x))$  will be respectively the following formulas:

- $\Phi_C(F(x))$  is:  $\forall x_1 D_1(x_1), \dots, \forall x_k D_k(x_k) \rightarrow F[t/x]$ ;
- $\Psi_C(F(x))$  is:  $\forall x_1 D_1(x_1), \dots, \forall x_k D_k(x_k), \forall z (D z_{\rightarrow t} \rightarrow F[z/x]) \rightarrow F[t/x]$ .

The notation may suggest some kind of formulas that are actually useful in the construction of I-proofs which are defined as follows:

**Definitions 2.5. [I-formulas and restrictive hypothesis]**

- A formula  $F$  is called an *I-formula* if and only if  $F$  is of the form  $H_1, \dots, H_m \rightarrow D(f(t_1, \dots, t_n))$  for some:
  - data type  $D$ , defined function  $f$ ,
  - formulas  $H_i$  for  $i = 1, \dots, m$  such that  $H_i$  is of the form  $\forall x D'(x)$  or of the form  $\forall z (D' z_{\rightarrow u} \rightarrow F')$  for some data type  $D'$ , I-formula  $F'$  and term  $u$ .
- An *I-restrictive hypothesis* of an I-formula  $F$  of the form  $H_1, \dots, H_m \rightarrow D(f(t_1, \dots, t_n))$  is a formula  $H_i$  of the form  $\forall z (D' z_{\rightarrow u} \rightarrow F')$ . We say that  $H'$  is a *restrictive hypothesis* to an I-restrictive hypothesis  $H = \forall z (D' z_{\rightarrow u} \rightarrow F')$  if  $H'$  is an I-restrictive hypothesis of the I-formula  $F'$ .

The definition of an I-formula is recursive, and an I-formula may have sub-I-formulas. An I-restrictive hypothesis is not an I-formula and we can use the term *restrictive hypothesis* to also denote I-restrictive hypothesis. The termination statement of a defined function is an I-formula which has no restrictive hypothesis.

The lemmas below state that one can use two additional rules, called *Struct* rule and *Ind* rule, in *TTR* as they can be derived from the other rules of *TTR*. These rules correspond to macro-rules, the former one can be seen as a reasoning by cases, while the last one stands for an induction rule.

**Lemma-Definition 2.6. [The Ind rule]**

Let  $D$  be a data type and consider all the constructor functions  $C_i$  of type  $D_{i_1}, \dots, D_{i_k} \rightarrow D$ ,  $0 \leq i_k, i = 1, \dots, q$ . Let  $P$  be a formula of the form  $F_1, \dots, F_k, \forall x D(x), F_{k+1}, \dots, F_m \rightarrow D'(t)$ , and  $\Gamma$  a context. For  $\Psi_{C_i}(P_{-D(x)})$  given as in Notation 2.4, the induction *Ind* rule on type  $D$  is:

$$\frac{\Gamma \vdash_{\mathcal{E}} \Psi_{C_1}(P_{-D(x)}) \quad \dots \quad \Gamma \vdash_{\mathcal{E}} \Psi_{C_q}(P_{-D(x)})}{\Gamma \vdash_{\mathcal{E}} P} \quad \text{Ind}(x)$$

Along with the *Ind* rule, the *Struct* rule defined below, which is also a macro-rule derived from *TTR*, can be considered as a reasoning by cases.

**Lemma-Definition 2.7.** [The *Struct* rule]

Let  $D$  be a data type and consider all the constructor functions  $C_i$  of type  $D_{i_1}, \dots, D_{i_k} \rightarrow D$ ,  $0 \leq i_k$ ,  $i = 1, \dots, q$ . Let  $P$  be a formula of the form  $F_1, \dots, F_k, \forall x D(x), F_{k+1}, \dots, F_m \rightarrow D'(t)$ , and  $\Gamma$  a context. For  $\Phi_{C_i}(P_{-D(x)})$  given as in Notation 2.4, the *Struct* rule on type  $D$  is:

$$\frac{\Gamma \vdash_{\mathcal{E}} \Phi_{C_1}(P_{-D(x)}) \quad \dots \quad \Gamma \vdash_{\mathcal{E}} \Phi_{C_q}(P_{-D(x)})}{\Gamma \vdash_{\mathcal{E}} P} \quad \text{Struct}(x)$$

Due to these lemmas, two macro-rules can be added in *TTR*: the *Struct*-rule (Lemma 2.7) and the *Ind*-rule (Lemma 2.6). From these rules, *distributing trees* can be built in *ProPre* (see Definition 2.10).

**Remark 2.8.** I-formulas are preserved by the *Struct*-rule and the *Ind*-rule. That is, if  $P$  is an I-formula, then so are:  $\Phi_C(P_{-D(x)})$  and  $\Psi_C(P_{-D(x)})$ .

**Definition 2.9.** [Heart of formula] The *heart* of a formula of the form  $F = H_1, \dots, H_m \rightarrow D(t)$ , where  $D$  is a recursive data type, will be the term  $t$ , denoted by  $H(F)$ .

The distributing trees are defined as follows:

**Definition 2.10.** [Distributing tree] Assume  $\mathcal{E}_f$  is a specification of a function  $f : D_1, \dots, D_n \rightarrow D$ .  $\mathcal{A}$  is a *distributing tree* for  $\mathcal{E}_f$  iff  $\mathcal{A}$  is a proof tree built only with the *Struct* rule and *Ind* rule where:

1. the root of  $\mathcal{A}$  is the termination statement of  $f$  with the empty context, i.e.:  
 $\vdash_{\mathcal{E}_f} \forall x_1 D_1(x_1), \dots, \forall x_n D_n(x_n) \rightarrow D(f(x_1, \dots, x_n))$ .
2. if  $\mathcal{L} = \{\Gamma_1 \vdash_{\mathcal{E}_f} F_1, \dots, \Gamma_q \vdash_{\mathcal{E}_f} F_q\}$  is the set of  $\mathcal{A}$ 's leaves, then there exists a one to one application  $\mathcal{B}: \mathcal{L} \hookrightarrow \mathcal{E}_f$  such that  $\mathcal{B}(L) = (t, u)$  with  $L = (\Gamma \vdash_{\mathcal{E}_f} F)$  in  $\mathcal{L}$  and the heart of  $F$  is  $H(F) = t$ .

Note that it can be inductively checked, from the root, using remark 2.8, that any formula in a distributing tree is an I-formula.

The I-proofs found by the *ProPre* system are formal termination proofs of termination statements of defined functions. They are divided into three phases:

1. the development of a distributing tree for the specification of a defined function, characterized by a property, called formal terminal state property;
2. each leaf of the distributing tree is extended into a new leaf by an application of an (*eq*) rule;
3. each leaf, coming from the second step, is extended with a new sub-tree, with the use of rules defined in [12], whose leaves end with axiom rules.

Due to the following fact proved in [12], it is not necessary to consider in this paper the middle and upper parts of proof trees built in the *ProPre* system:

**Fact 2.11.** A distributing tree  $T$  can be (automatically) extended into a complete proof tree iff  $T$  enjoys a property, called the *formal terminal state property*.

That is, it is enough to look at distributing trees that have the formal terminal state property to be able to complete the proof tree and hence state the termination of the function. Therefore it remains for us to state the mentioned property.

**Definition 2.12.** We say that an I-formula or a restrictive hypothesis  $P$  can be applied to a term  $t$  if the heart  $H(P)$  of  $P$  matches  $t$  according to a substitution  $\sigma$  where for each variable  $x$  that occurs free in  $P$  we have  $\sigma(x) = x$ .

The relation  $\prec$  of Definition 2.5 deals with the measure  $|\cdot|_{\#}$  on the terms, ranging over natural numbers, which counts the number of subterms of a given term  $t$  (including  $t$ ), and is interpreted as follows:

**Definition 2.13.** Let  $\mathcal{V}ar(t)$  be the set of variables occurring in  $t$ . Let  $u, v$  be terms. We say that  $u \prec v$  iff:  $|u|_{\#} < |v|_{\#}$ ,  $\mathcal{V}ar(u) \subseteq \mathcal{V}ar(v)$ , and  $u$  is linear.

This clearly defines a well-founded ordering  $\prec$  on terms. We can now state the main property that a distributing tree must enjoy in the I-proofs of *ProPre*.

**Definition 2.14. [Formal Terminal State Property]**

Let  $\mathcal{E}_f$  be a specification of a function  $f$  and  $\mathcal{A}$  be a distributing tree for  $\mathcal{E}_f$ . We say that  $\mathcal{A}$  satisfies the *formal terminal state property (ftsp)* iff for all leaves  $L = (\Gamma \vdash_{\mathcal{E}_f} F)$  of  $\mathcal{A}$  with the equation  $e \in \mathcal{E}_f$  such that  $\mathcal{B}(L) = e$ , where  $\mathcal{B}$  is the application given in Definition 2.10, and for all recursive calls  $(t, v)$  of  $e$ , there exists a restrictive hypothesis  $P = \forall z Dz_{\prec s}, H_1, \dots, H_k \rightarrow D(w)$  of  $F$  and a substitution  $\sigma$  such that  $P$  can be applied to  $v$  according to  $\sigma$  with:

1.  $\sigma(z) \prec s$  and
2. for all restrictive hypothesis  $H$  of  $P$  of the form  $\forall y D' y_{\prec s'} \rightarrow K$  there is a restrictive hypothesis  $H_0$  of  $F$  of the form  $\forall y D' y_{\prec s_0} \rightarrow K$  with  $\sigma(s') \preceq s_0$ .

So, *ProPre* establishes the termination of a function  $f$  by showing that the distributing tree of the specification of  $f$  (which is a partial tree whose root is the termination statement of  $f$ ) has the formal terminal state property (and hence can be extended into a complete proof tree of the termination statement of  $f$ ).

### 3 The abstract terminal state property

Proof structures can often be heavy and difficult to work with. However, in the constructive framework of the Curry-Howard isomorphism, compiling a recursive algorithm corresponds to establishing a formal proof of its totality. In *ProPre*, termination proofs play an important role as they make it possible to obtain  $\lambda$ -terms that compute programs. We set out to simplify the termination techniques developed in *ProPre* by showing that its automated formal proofs can be abstracted giving rise to a simpler property which respects termination. Instead of dealing with formulas, we will use the simpler concept of functions. Also, instead of data symbols, we will use sorts and assume that there is a correspondence between the data types of *ProPre* and our sorts. Instead of the complex concept of distributing trees used in *ProPre* (Definition 2.10), we will use the much simpler notion of *term distributing trees* of [14]. By living in the easier framework, we will introduce the new *abstract terminal state property* which will play for term distributing trees a similar role to that played by the formal terminal state property for distributing trees. In this section we present a data structure for which we will be able to introduce a new termination property.

We consider a countable set  $\mathcal{X}$  of individual variables and we assume that each variable of  $\mathcal{X}$  has a unique sort and that for each sort  $s$  there is a countable number of variables in  $\mathcal{X}$  of sort  $s$ . For sort  $s$ ,  $F$  subset of  $\mathcal{F}$ , and  $\mathcal{X}$  subset of  $\mathcal{X}$ ,  $\mathcal{T}(F, \mathcal{X})_s$  denotes the set of terms of sort  $s$  built from  $F$  and  $\mathcal{X}$ . In case  $\mathcal{X}$  is empty we will also use the notation  $\mathcal{T}(F)_s$ .

We recall the definition of *term distributing trees* of [14]. A term distributing tree is much simpler than the distributing tree of *ProPre* given in Definition 2.10. The novelty of

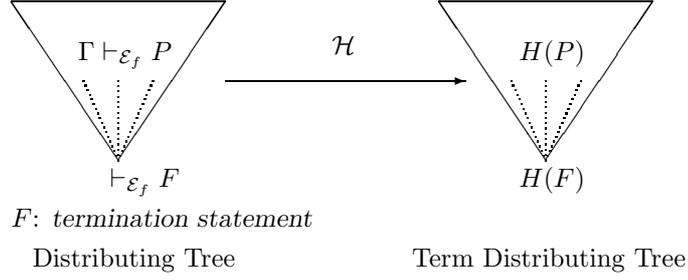


Figure 1: The operator  $\mathcal{H}$

this section will be a term distributing tree equipped with abstract terminal state property (Definition 3.5 below).

**Definition 3.1.** [Term distributing tree] Let  $\mathcal{E}_f$  be a specification of a function  $f : s_1, \dots, s_n \rightarrow s$ .  $T$  is a *term distributing tree* for  $\mathcal{E}_f$  iff it is a tree where:

1. its root is of the form  $f(x_1, \dots, x_n)$  where  $x_i$  is a variable of sort  $s_i$ ,  $i \leq n$ ;
2. each of its leafs is a left-hand side of an equation of  $\mathcal{E}_f$  (up to variable renaming); and
3. each node  $f(t_1, \dots, t_n)$  of  $T$  admits one variable  $x'$  of a sort  $s'$  such that the set of children of the node is  $\{f(t_1, \dots, t_n)[C(x'_1, \dots, x'_r)/x']\}$ , where  $x'_1, \dots, x'_r$  are not in  $t_1, \dots, t_n$  and  $C : s'_1, \dots, s'_r \rightarrow s' \in \mathcal{F}_c$ .

A term distributing tree can be seen as a skeleton form of a distributing tree  $T$  by taking the heart of the formulas in the nodes of  $T$ , which gives rise to an operator  $\mathcal{H}$  illustrated by Figure 1.

Therefore we have the following proposition:

**Proposition 3.2.** If there is a distributing tree for a specification  $\mathcal{E}_f$  of a function  $f$  then there is also a term distributing tree for the specification  $\mathcal{E}_f$ .

A term distributing tree is easier to handle than a distributing tree. But, in both parts of Figure 1, term distributing trees and distributing trees may have no termination property. However, we know by Fact 2.11 that a function terminates if we have a distributing tree that satisfies a right terminal state property. What we want is to define a notion on the term distributing trees that also ensures the termination of functions. We first give some notations and remarks.

**Notations 3.3.** Let  $T$  be a term distributing tree with root  $\theta_1$ .

- A branch  $b$  from  $\theta_1$  to a leaf  $\theta_k$  is denoted by  $(\theta_1, y_1), \dots, (\theta_{k-1}, y_{k-1}), \theta_k$  where for each  $i \leq k-1$ ,  $y_i$  corresponds to the variable  $x'$  for the node  $\theta_i$  in the third clause of Definition 3.1. We use  $L_b$  to denote the leaf of the branch  $b$ .
- If a node  $\theta$  matches a term  $u$  of a recursive call  $(t, u)$  then the substitution will be denoted by  $\rho_{\theta, u}$  (in particular in Definition 3.5).
- For a term  $t$  of a left-hand side of an equation,  $b(t)$  will denote the branch in the term distributing tree that leads to  $t$  (second clause of Definition 3.1).

**Remarks 3.4.**

- Let  $f : s_1, \dots, s_n \rightarrow s$  be a function and  $\mathcal{E}_f$  be a specification of  $f$ . Let  $T$  be a term distributing tree of  $\mathcal{E}_f$ . Then for each  $(w_1, \dots, w_n)$  of  $\mathcal{T}(\mathcal{F}_c)_{s_1} * \dots * \mathcal{T}(\mathcal{F}_c)_{s_n}$  there is one and only one leaf  $\theta$  of  $T$  and a ground constructor substitution  $\varphi$  such that  $\varphi(\theta) = f(w_1, \dots, w_n)$ .
- Let  $T$  be a term distributing tree for a specification and let  $b$  be a branch from the root  $\theta_1$  of  $T$  to a leaf  $\theta_k$  with  $b = (\theta_1, x_1), \dots, (\theta_{k-1}, x_{k-1}), \theta_k$ . Then for each node  $\theta_i, \theta_j$  with  $1 \leq i \leq j \leq k$ , there exists a constructor substitution, denoted  $\sigma_{\theta_j, \theta_i}$ , such that  $\sigma_{\theta_j, \theta_i}(\theta_i) = \theta_j$ .

Now, we give the *abstract terminal state property* for term distributing trees:

**Definition 3.5. [Abstract terminal state property]**

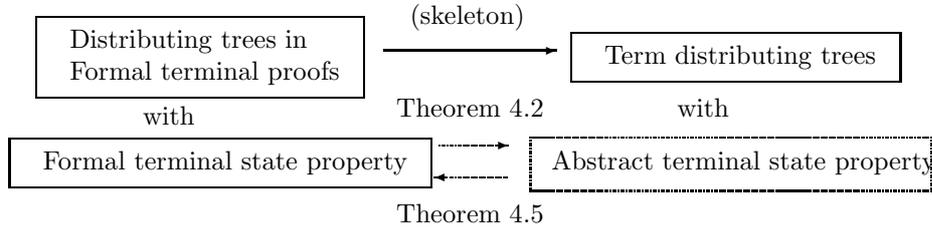
Let  $T$  be a term distributing tree for a specification. We say that  $T$  has the *abstract terminal state property* (*atsp*) if there is an application  $\mu : T \rightarrow \{0, 1\}$  on the nodes of  $T$  such that if  $L$  is a leaf,  $\mu(L) = 0$ , and for every recursive call  $(t, u)$ , there is a node  $(\theta, x)$  in the branch  $b(t)$  with  $\mu(\theta) = 1$  such that  $\theta$  matches  $u$  with  $\rho_{\theta, u}(x) \prec \sigma_{L_{b(t)}, \theta}(x)$  (cf. Notations 3.3 and Remark 3.4) and for all ancestors  $(\theta', x')$  of  $\theta$  in  $b(t)$  with  $\mu(\theta') = 1$ , we have  $\rho_{\theta', u}(x') \preceq \sigma_{L_{b(t)}, \theta'}(x')$ .

Note that similarly to term distributing trees, no formula is mentioned in the definition of *atsp* and hence *atsp* is easier to handle than *ftsp* (Definition 2.14) because *atsp* only uses relations of substitutions where all proposition informations have been abstracted. However, it is not obvious that a term distributing tree that satisfies *atsp* implies the termination of the given function. A way to prove this fact would be to infer some particular measures from such distributing trees and to show that these measures have the decreasing property through the recursive calls of the given function so that the function terminates.

We will not follow this way because we want to prove a stronger result: we will show in the next section that from a term distributing tree that has the *atsp* we can reconstruct an I-proof, which implies that the given function terminates and also enables a  $\lambda$ -term that computes the function to be extracted.

## 4 Building formal proofs from skeleton forms

The aim of this section is first to show that the *atsp* can be viewed as an abstract form of the *ftsp*. This is formally stated with Theorem 4.2 below. Secondly, Theorem 4.5 states that the *atsp* is a sufficient condition to construct a distributing tree with the *ftsp* from a term distributing tree (skeleton form). This can be illustrated with the picture below.



We start by extending the application  $\mathcal{H}$  (Figure 1) into a new operator  $\mathcal{H}'$  from a distributing tree  $\mathcal{A}$  to the term distributing tree  $\mathcal{H}(\mathcal{A})$  which is now equipped with an application  $\mu : \mathcal{H}(\mathcal{A}) \rightarrow \{0, 1\}$  defined on the node of  $\mathcal{H}(\mathcal{A})$ , so that  $\mathcal{H}'(\mathcal{A})$  will be  $(\mathcal{H}(\mathcal{A}), \mu)$ . A term distributing tree equipped with an application  $\mu$  will also be called a  $\mu$ -term distributing tree.

To define the operator  $\mathcal{H}'$ , the application  $\mu$  is given as follows: Let  $\mathcal{A}$  be a distributing tree and  $(\Gamma \vdash_{\mathcal{E}_f} P)$  be a node of  $\mathcal{A}$ . If  $(\Gamma \vdash_{\mathcal{E}_f} P)$  is a leaf, we take  $\mu(H(P)) = 0$ . If not, we consider  $\mu(H(P)) = 1$  if the rule applied on  $(\Gamma \vdash_{\mathcal{E}_f} P)$  in  $\mathcal{A}$  is the *Ind* rule and  $\mu(H(P)) = 0$  otherwise.

Note that  $\mathcal{H}$  is not injective: there is at least two distinct distributing trees  $\mathcal{A}$  and  $\mathcal{A}'$  such that  $\mathcal{H}(\mathcal{A}) = \mathcal{H}(\mathcal{A}')$ . However,  $\mathcal{H}'$  is injective. Actually if we consider term distributing trees equipped with a  $\mu$ -application, then  $\mathcal{H}'$  becomes bijective and the inverse operator of  $\mathcal{H}'$  can be stated with the definition below.

**Lemma-Definition 4.1.** [ $\mathcal{D}$ , the inverse of  $\mathcal{H}'$ ] Let  $\mathcal{E}_f$  be a specification of a function  $f : s_1, \dots, s_n \rightarrow s$ , and let  $(T, \mu)$  be a term distributing tree for  $\mathcal{E}_f$  (equipped with a  $\mu$  application). There is one and only one distributing tree  $\mathcal{A}$  for  $\mathcal{E}_f$  such that  $\mathcal{H}'(\mathcal{A}) = (T, \mu)$ . This one can be automatically obtained from  $(T, \mu)$  and we define the application  $\mathcal{D}$  with  $\mathcal{D}(T, \mu) = \mathcal{A}$ .

**Proof:** Let  $F = \forall x_1 D_1(x_1), \dots, \forall x_n D_n(x_n) \rightarrow D(f(x_1, \dots, x_n))$  be the termination statement of  $f$ . We can inductively *build* a distributing tree  $\mathcal{A}$  of the same size as  $T$  by taking the root of  $\mathcal{A}$  to be  $\vdash_{\mathcal{E}_f} F$  and assuming the existence of a node  $(\Gamma \vdash_{\mathcal{E}_f} P)$  of  $\mathcal{A}$ , for  $P$  is an I-formula, such that:

- i)  $P$  is of the form:  $F_1, \dots, F_r, \forall x D'(x), F_{r+1}, \dots, F_p \rightarrow D(f(t_1, \dots, t_n))$  where  $D$  and  $D'$  are data symbols, and variables in the heart of  $P$  are bound,
- ii)  $T$  admits a level, the same as those  $(\Gamma \vdash_{\mathcal{E}_f} P)$  in  $\mathcal{A}$ , such that the node  $\theta$  at this level is distinct from a leaf, with  $\theta = f(t_1, \dots, t_n)$  whose variable according to Definition 3.1.3 is the variable  $x$  of sort  $s'$  associated to  $D'$ .

From above, we build the children nodes of  $(\Gamma \vdash_{\mathcal{E}_f} P)$  in  $\mathcal{A}$  as follows:

- If  $\mu(\theta) = 0$ , the node  $(\Gamma \vdash_{\mathcal{E}_f} P)$  is extended by the *Struct* rule on  $x$  in  $P$ .
- If  $\mu(\theta) = 1$ , the node  $(\Gamma \vdash_{\mathcal{E}_f} P)$  is extended using the *Ind* rule on  $x$  in  $P$ .

In both cases, since  $P$  is an I-formula, if  $P'_j$  denotes either  $\Psi_{C_j}(P_{-D(x)})$  or  $\Phi_{C_j}(P_{-D(x)})$  of Definitions 2.6 and 2.7 as a children node of  $P$ , then  $P'_j$  is an I-formula. As the variables that occur in  $P$  are bound, by construction of its children, the variables occurring in the heart of  $P'_j$  are bound too. Now, due to the definitions of the term distributing trees and the *Ind* and *Struct* rules, it is easy to see that there is a child node  $\theta_j$  of  $\theta$  such that  $C(P'_j) = \theta'_j$ . Therefore, the above process allows the property ii) to be held by each child of  $(\Gamma \vdash_{\mathcal{E}_f} P)$  except if the corresponding node in  $T$  is a leaf. By definition of  $\mathcal{A}$ ,  $\mathcal{C}'(\mathcal{A}) = (T, \mu)$  and its uniqueness results from injectivity of  $\mathcal{C}'$ . This gives the associated tree  $\mathcal{A} = \mathcal{D}(T)$  of  $T$  with  $\mathcal{C}'(\mathcal{D}(T, \mu)) = (T, \mu)$ . Hence we deduce, because  $\mathcal{C}'$  is injective, that  $\mathcal{D}(\mathcal{C}'(\mathcal{A})) = \mathcal{A}$  for each distributing tree.  $\square$

This means that for any distributing tree  $\mathcal{A}$  and term distributing tree  $(T, \mu)$ , we have:  $\mathcal{D}(\mathcal{H}'(\mathcal{A})) = \mathcal{A}$  and  $\mathcal{H}'(\mathcal{D}(T, \mu)) = (T, \mu)$ . We can illustrate  $\mathcal{D}$  with Figure 2.

However there is still no warranty on the termination of functions using  $\mu$ -term distributing trees. The first theorem below shows that the *atsp* of  $\mu$ -term distributing trees stands in some sense for the *ftsp* from which all proposition informations are abstracted in a simpler context.

**Theorem 4.2.** Let  $\mathcal{E}_f$  be a specification of a function  $f$  and  $\mathcal{A}$  be a distributing tree for  $\mathcal{E}_f$ . If  $\mathcal{A}$  has the formal terminal state property then the term distributing tree  $\mathcal{H}'(\mathcal{A})$  has the abstract terminal state property.

**Proof:** Similar to the proof of Theorem 4.5 below.  $\square$

Before giving the opposite of Theorem 4.2, Theorem 4.5, we need to introduce the following two definitions:

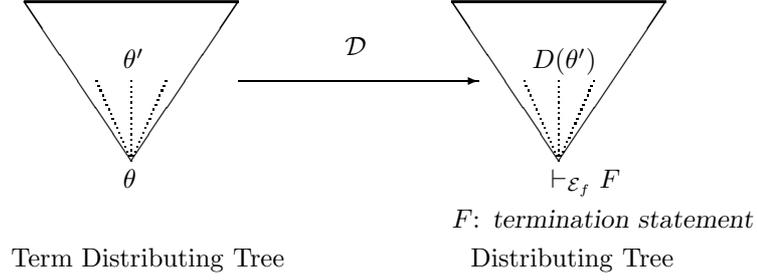


Figure 2: The reverse operator of  $\mathcal{H}'$

**Definition 4.3.**  $[N_r(Q, P)]$  Let  $P$  be an I-formula and  $Q$  a restrictive hypothesis of  $P$ .  $N_r(Q, P)$  is the number of restrictive hypotheses of  $P$  that appear between the outermost restrictive hypothesis of  $P$ . E.g., if  $Q$  is the outermost restrictive hypothesis of  $P$ , then  $N_r(Q, P) = 1$ .  $N_i(P)$  is the number of restrictive hypothesis of  $P$ .

**Definition 4.4.**  $[Tr_b^{j,k}(Q)]$  Let  $\mathcal{A}$  be a distributing tree for a specification  $\mathcal{E}_f$ . Let  $b$  be a branch and  $P$  a node in  $b$  at a level  $i$  from the root. We define  $Tr_b^{i+1,i}(Q)$ , where  $Q$  is a restrictive hypothesis of  $P$ , as the restrictive hypothesis  $Q'$  in the child  $P'$  of  $P$  in  $b$  as follows depending on whether the rule applied on  $P$  is:

- *Struct*:  $Q'$  is the restrictive hypothesis where  $N_r(Q', P') = N_r(Q, P)$ .
- *Ind*:  $Q'$  is such that  $N_r(Q', P') = N_r(Q, P) + 1$ .

We also define  $Tr_b^{j,k}(Q)$  with  $j > k$  as the restrictive hypothesis of the node  $P''$  at level  $j$  in  $b$  defined by:  $Tr_b^{j,k}(Q) = Tr_b^{j,j-1} \circ \dots \circ Tr_b^{k+2,k+1} \circ Tr_b^{k+1,k}(Q)$ .

Finally  $Tr_b^{i,i}$  will denote the identity on  $P$ .

The next theorem is the opposite of Theorem 4.2 and shows that we can automatically rebuild a distributing tree that has the *ftsp* from a *skeleton form* that has the *atsp*. As a consequence, according to Section 2.3, we can also build an I-proof and thus extract a  $\lambda$ -term that computes the given function.

**Theorem 4.5.** Let  $\mathcal{E}_f$  be a specification of a function  $f$  and  $(T, \mu)$  be a  $\mu$ -term distributing tree for  $\mathcal{E}_f$ . If  $(T, \mu)$  has the abstract terminal state property then the distributing tree  $\mathcal{D}(T, \mu)$  has the formal terminal state property.

**Proof:** Let  $(T, \mu)$  be a term distributing tree for  $\mathcal{E}_f$  which has the *Atsp*. We want to show that  $\mathcal{D}(T, \mu)$  has the *ftsp*. Take a recursive call  $(t, v)$  of an equation of  $\mathcal{E}_f$ . We have to find a restrictive hypothesis  $R = \forall z Dz \prec_s, F_1, \dots, F_k \rightarrow D(w)$  in  $L$  of  $\mathcal{D}(T, \mu)$  with  $\mathcal{B}(L) = (t, v)$ , where  $\mathcal{B}$  is the application of Definition 2.10, such that clauses 1. and 2. of Definition 2.14 hold. Let  $B$  be the corresponding branch in  $\mathcal{D}(T, \mu)$  of  $b(t)$  in  $T$ , and let  $(\theta, x)$  be the node in  $b(t)$  given in Definition 3.5. Consider  $(\Gamma \vdash_{\mathcal{E}_f} P)$  in  $\mathcal{D}(T, \mu)$  that is at the same level of  $(\theta, x)$  in  $T$ . As  $\mu(\theta) = 1$ , by construction of  $\mathcal{D}(T, \mu)$ , a new restrictive hypothesis of the form  $Q = \forall z (Dz \prec_s \rightarrow P_{-D(x)}[z/x])$  is created in the child  $P'$  of  $P$  in  $B$ . Consider  $R = Tr_B^{j,i}(Q)$  the restrictive hypothesis in  $B$  where  $i$  and  $j$  are respectively the level of  $P'$  and the leaf of  $B$ . We can write  $R = \forall z (Dz \prec_{s'} \rightarrow P_{-D(x)}[z/x])$  for some term  $s'$  because:

- 1) The free variables in  $Q$  are those of the term  $s$ , and the applied *Ind/Struct* rule is done on a variable in  $P'$  which is out of the scope of  $Q$ .

- 2) As 1) first holds for  $Q' = Tr_B^{i+1,i}(Q)$ , next holds for  $Tr_B^{i+2,i}(Q) = Tr_B^{i+2,i+1}(Q')$ ,  $\dots$ , we have that:  $R = Tr_B^{j,i}(Q) = \forall z(Dz \prec_{s'} \rightarrow P_{-D(x)}[z/x])$  where the variables of  $C(R)$  are closed in  $R$ .

**Clause 1** We know that  $\theta$  matches  $v$  with a substitution  $\rho_{\theta,v}$ , but  $C(P) = \theta$ , so  $R$  can be applied to  $v$  according to a substitution  $\sigma$  defined with  $\sigma(z) = \rho_{\theta,v}(x)$  and  $\sigma(y) = \rho_{\theta,v}(y)$  for  $y \neq z$ . We have to show that  $\sigma(z) \prec s'$ . This can be easily proved, by induction on  $k \geq i$ , that if  $Tr_B^{k,i}(Q) = \forall z(Dz \prec_{s_k} \rightarrow P_{-D(x)}[z/x])$  for some term  $s_k$ , then  $s_k = \sigma_{k,i-1}(x)$  where the node  $\theta$  matches the node at level  $k$  in  $T$  with the substitution  $\sigma_{k,i-1}$ . By definition of  $j$ ,  $\sigma_{j,i-1} = \sigma_{L_B,\theta}$ , so  $\rho_{\theta,v}(x) \prec \sigma_{j,i-1}(x)$  by Definition 3.5, and we can now deduce that  $\sigma(z) \prec s'$  since  $s' = s_j$ . Therefore clause 1. of Definition 2.14 holds.

**Clause 2** Consider a restrictive hypothesis  $H = \forall zD'z \prec_r \rightarrow K$  in  $R$ ; we have to find a restrictive hypothesis  $H_0$  in  $P$  of the form  $\forall zD'z \prec_{r_0} \rightarrow K$  such that  $\sigma(r) \preceq r_0$ . As  $H$  is a restrictive hypothesis of  $Tr_B^{j,i}(Q)$ ,  $H$  is also a restrictive hypothesis of  $Q$ . Hence, one associates to  $H$  a restrictive hypothesis  $H'$  in

$$P' = \forall x_{i_1} D_{i_1}(x_{i_1}), \dots, \forall x_{i_k} D_{i_k}(x_{i_k}), \underbrace{\forall z(Dz \prec_{s_i} \rightarrow P_{-D(x)}[z/x])}_Q \rightarrow P_{-D(x)}[s_i/x],$$

where  $H$  and  $H'$  respectively appear in  $P_{-D(x)}[z/x]$  and  $P_{-D(x)}[s_i/x]$ . As  $H$  is of the form  $\forall zD'z \prec_r \rightarrow K$  then  $H'$  is of the form  $\forall zD'z \prec_{r_0} \rightarrow K$  since only the variables in the term  $r$  are free in  $H$ . Now consider the node  $(\Gamma \vdash_{\mathcal{E}_f} N)$  in  $B$  at a level  $l$  such that 1) a new restrictive hypothesis  $M$  is created in the child  $N'$  of  $N$  in  $B$ , namely,  $N_i(N') = N_i(N) + 1$  and  $N_r(M, N') = 1$ , and 2)  $Tr_B^{i,l}(M) = H'$ . Let  $(\theta', x')$  be the corresponding node in  $T$  of  $(\Gamma \vdash_{\mathcal{E}_f} N)$  in  $A$ . It is clear that  $\theta'$  is an ancestor of  $\theta$  in  $T$  since  $l < j$  in  $\mathcal{D}(T, \mu)$ . Furthermore as  $N_i(N') = N_i(N) + 1$ , we have  $\mu(\theta') = 1$ . By Definition 3.5 we have the relation  $\rho_{\theta',v}(x') \preceq \sigma_{L_{b(i)},\theta'}(x')$ . Let us now choose  $H_0 = Tr_B^{j,l+1}(M)$  as the restrictive hypothesis in  $P'$ . Using the same property of clause 1 as we did with  $Tr_B^{j,i}(Q)$ , we know that  $r_0$  is  $\sigma_{j,l}(x') = \sigma_{L_{b(i)},\theta'}(x')$ . Let us show that  $\sigma(r) = \sigma_{\theta',v}(x')$ . We note that  $i - 1 \geq l + 1$  since  $i - 1$  and  $l$  are respectively the level of  $P$  and  $N$  that are distinct. We have  $Tr_B^{i-1,l+1}(M) = \forall z(D'z \prec_{\sigma_{i-1,l}(x')} \rightarrow K)$  in  $P$ , where  $\sigma_{i-1,l}$  is by definition the substitution  $\sigma_{\theta,\theta'}$ . So, according to the restrictive hypothesis  $Q$  in  $P'$ , the term  $r$  in  $H$  is  $\sigma_{\theta,\theta'}(x')[z/x]$ . Now, by definition of  $\sigma$  in clause 1 of Definition 2.14, we have  $\sigma(r) = \rho_{\theta,v} \{z \rightarrow x\}(\sigma_{\theta,\theta'}(x')[z/x]) = \rho_{\theta,v}(\sigma_{\theta,\theta'}(x'))$ . But the relation of substitutions gives us  $\rho_{\theta',v} = \rho_{\theta,v} \circ \sigma_{\theta,\theta'}$ . So we finally obtain  $\sigma(r) = \rho_{\theta',v}(x')$ , and we can deduce from the above and Definition 3.5 that  $\sigma(r) \preceq r_0$ . Hence, clause 2. of Definition 2.14 holds.  $\square$

In [5], measures were related to given functions whose decreasing property through the recursive calls were dependent on the *ftsp* enjoyed by distributing trees. We claim that it is possible to infer measures directly from term distributing trees whose decreasing property through the recursive calls of the considered functions now rely only on *atsp*. We do not state the measures for lack of space but just remark that this is a straightforward consequence of the results of this section with the previous one and [5].

Following distributing tree with *atsp* makes the analysis of the I-proofs easier. In particular there are no measures from [5] associated to the *quot* function (defined in the next section) that have the decreasing property (see [4]). As a consequence of the above results of this section, there are no I-proofs for such function. The aim of the following section is to show that the framework of *ProPre* can actually be applied to new functions (e.g. *quot* function) provided an automated termination procedure (e.g. [4, 1, 2]) is used.

## 5 Synthesizing programs from termination techniques

As noted in Section 2, if we can prove, in *TTR*, a *formula* that states the totality of a function then it is possible, in term of programs, to obtain a  $\lambda$ -term as the code of the

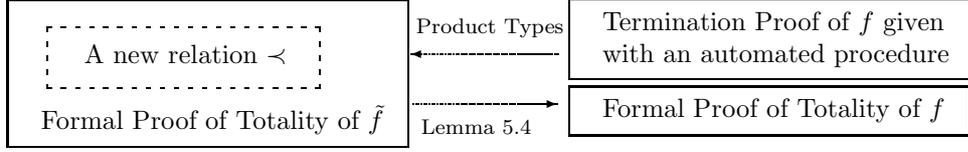


Figure 3: A formal proof of totality of the function  $f$ .

function. As earlier mentioned, this formula is called *termination statement* in *ProPre* (Definition 2.2). More precisely, assume that  $\mathcal{E}_{f_1}, \dots, \mathcal{E}_{f_m}$  are specifications of defined functions already proved in the *ProPre* system. Let  $f$  be a new defined function with a specification  $\mathcal{E}_f$ . We put  $\mathcal{E} = \sqcup_{j=1}^{j=n} \mathcal{E}_{f_j}$ , and  $\mathcal{E}_f^1 = \mathcal{E}_f \sqcup \mathcal{E}$ . In order to obtain a lambda term  $F$  that computes the new function  $f$ , *ProPre* needs to establish  $\vdash_{\mathcal{E}_f^1} F : T_f$  in *TTR*.

Now consider the following specification function :

**Example 5.1.** Let  $quot : nat, nat, nat \rightarrow nat$  be a defined function with specification  $\mathcal{E}_{quot}$  given by the equations:

$$\begin{array}{ll} quot(x, 0, 0) = 0 & quot(s(x), s(y), z) = quot(x, y, z) \\ quot(0, s(y), z) = 0 & quot(x, 0, s(z)) = s(quot(x, s(z), s(z))) \end{array}$$

The value  $quot(x, y, z)$  corresponds to  $1 + \lfloor \frac{x-y}{z} \rfloor$  when  $z \neq 0$  and  $y \leq x$ , that is to say  $quot(x, y, y)$  computes  $\lfloor \frac{x}{y} \rfloor$ . Its specification does not admit an I-proof and therefore no  $\lambda$ -term can be associated by the *ProPre* system.

To circumvent this drawback, we show, considering the framework of *ProPre* and *TTR*, that it is possible to add other automated termination procedures than the one of *ProPre* regarding the automation of the extraction of  $\lambda$ -terms.

When *ProPre* builds a formal proof of a specification, it needs to check at different steps that some subterm in one argument of the equations *decreases* in the recursive calls according to the relation devised in Definition 2.13. These informations are given by a termination algorithm designed in *ProPre*. Said informally, to convey the termination informations in the formal proof in *ProPre*, it is used with the relation  $\prec$  included in formulas of the form  $A[u/x] \uparrow (u \prec v)$  due to Table 2.

Now assume, for a given function that terminates, the equations admit only one argument. This provides a natural (partial) relation on the data type on which the function is specified so that each recursive call decreases. Also assume that an automated procedure ensures the termination of this function. Then this one can be used as the termination algorithm of *ProPre*, but we now consider the new relation instead of the earlier relation  $\prec$  of *ProPre*. Due to the hiding rules of the operator  $\uparrow$  we can develop a particular formal proof, as an I-Proof, for the considered function but where in particular the sequent  $\Gamma \vdash_{\mathcal{E}} (u \prec v)$  in the rule ( $\uparrow_1$ ) with  $e = (u \prec v)$  can be obtained with the new termination procedure that provides the new relation  $\prec$ .

In case the function admits several arguments, we would like to cluster the arguments of the equations of the specification into one argument. To do so, we show that the use of uncurryfication forms of functions is harmless in *TTR* (also in *AF2*) in the sense given by Lemma 5.4 by considering the product types. As a consequence this enables us to follow the principle illustrated in Figure 3 where  $\tilde{f}$  stands for an uncurryfication form of a given function  $f$ . The left part of Figure 3 is obtained with Theorem 5.7.

We will now come into more details to get the synthesis of a function concerning the above principle.

## 5.1 Product types

We introduce particular specifications that correspond in some sense to uncurryfication forms of earlier specifications. To do so, we will consider a product type associated to a function. As we have not stated the data types of  $TTR$  with the operator  $\mu$  (cf. beginning of Section 2.2), for the sake of presentation, we present below the product types in the context of  $AF2$ . This presentation in Definition 5.2 is harmless because Lemma 5.4 below and its proof hold both in  $AF2$  and  $TTR$ .

### Definition 5.2. [Product type of a function]

Let  $f : D_1, \dots, D_n \rightarrow D$  be a defined function,  $cp \in \mathcal{F}_c$  be a new constructor of arity  $n$  and take the termination statement of  $f$ :

$T_f = \forall x_1 \dots \forall x_n (D_1(x_1), \dots, D_n(x_n) \rightarrow D(f(x_1, \dots, x_n)))$ . The data type  $K(x)$  defined by the formula:  $\forall X \forall y_1 \dots \forall y_n (D_1(y_1), \dots, D_n(y_n) \rightarrow X(cp(y_1, \dots, y_n)) \rightarrow X(x))$  is called the product type of  $D_1, \dots, D_n$ , and is denoted by  $(D_1 \times \dots \times D_n)(x)$ .

Starting from the specification of a defined function  $f$  it is possible to associate another defined function  $\tilde{f}$  whose specification  $\mathcal{E}_{\tilde{f}}$  takes into account the product type of  $f$ .

**Definition 5.3.** Let  $f : D_1, \dots, D_n \rightarrow D$  be a defined function with a specification  $\mathcal{E}_f$ . Let  $\tilde{f}$  be a new defined symbol in  $\mathcal{F}_d$ , which is called the *twin function* of  $f$ . To define the specification  $\mathcal{E}_{\tilde{f}}$  of  $\tilde{f}$ , we define each equation  $\tilde{f}(cp(t_1, \dots, t_n)) = \bar{v}$  of  $\mathcal{E}_{\tilde{f}}$  from each equation  $f(t_1, \dots, t_n) = v$  of  $\mathcal{E}_f$  where  $cp$  is the constructor symbol of the product type associated to  $f$ . The term  $\bar{v}$  is recursively defined from  $v$  as follows:

- (i) if  $v$  is a variable or a constant then  $\bar{v} = v$ ,
- (ii) if  $v = g(u_1, \dots, u_m)$  with  $g$  a constructor or a symbol function distinct from  $f$ , then  $\bar{v} = g(\bar{u}_1, \dots, \bar{u}_m)$ ,
- (iii) if  $v = f(u_1, \dots, u_n)$  then  $\bar{v} = \tilde{f}(cp(\bar{u}_1, \dots, \bar{u}_n))$ .

Note that this clearly defines the specification  $\mathcal{E}_{\tilde{f}}$  of the defined function  $\tilde{f}$  associated to  $f$ , and that the termination statement of  $\tilde{f}$  is:

$$T_{\tilde{f}} = \forall x ((D_1 \times \dots \times D_n)(x) \rightarrow D(\tilde{f}(x))).$$

Let us consider the specification  $\mathcal{E}_f$  of a function and the set of equations  $\mathcal{E}'_f = \mathcal{E}_f \cup \{f(x_1, \dots, x_n) = \tilde{f}(cp(x_1, \dots, x_n))\}$ . The set  $\mathcal{E}'_f$  is not a specification according to Definition 2.2 in *ProPre*, but we can still reason in  $TTR$ . Assume the termination statement of  $\tilde{f}$  proved in  $TTR$  with  $\mathcal{E}_{\tilde{f}}$  and the set  $\mathcal{E}$  of the specifications already proved. Now we can add the equations of  $\mathcal{E}'_f$  in the set  $\mathcal{E}$  before proving the termination statement  $T_f$ . Due to the form of the specifications  $\mathcal{E}_{\tilde{f}}$  and  $\mathcal{E}_f$ , the equation  $f(x_1, \dots, x_n) = \tilde{f}(cp(x_1, \dots, x_n))$  does not add any contradiction in the set of the equational axioms  $\mathcal{E}_f \sqcup \mathcal{E}$ . Therefore we can now use the new set  $\mathcal{E}'_f \sqcup \mathcal{E}$  to prove the termination statement  $T_f$  in  $TTR$ . So, the equation  $f(x_1, \dots, x_n) = \tilde{f}(cp(x_1, \dots, x_n))$  provides the connection between  $\mathcal{E}_f$  and  $\mathcal{E}_{\tilde{f}}$  from the logical point of view and the proof of  $T_{\tilde{f}}$  provides the computational aspect of the function  $f$ . More precisely we have the following lemma.

**Lemma 5.4.** Let  $f : D_1, \dots, D_n \rightarrow D$  be a defined function with a specification  $\mathcal{E}_f$ , and  $\mathcal{E}_{\tilde{f}}$  the specification of the twin function  $\tilde{f}$ . Let  $\mathcal{E}_1, \dots, \mathcal{E}_n$  be the specifications of the defined functions already proved (in  $AF2$  or  $TTR$ ),  $\mathcal{E} = \sqcup_{i=1}^n \mathcal{E}_i$ . Let us note  $\mathcal{E}_{\tilde{f}}^1 = \mathcal{E}_{\tilde{f}} \sqcup \mathcal{E}$  and  $\mathcal{E}_{\tilde{f}}^2 = \mathcal{E}'_f \sqcup \mathcal{E}_{\tilde{f}}^1$  with  $\mathcal{E}'_f = \mathcal{E}_f \cup \{f(x_1, \dots, x_n) = \tilde{f}(cp(x_1, \dots, x_n))\}$ . If there is a  $\lambda$ -term  $\tilde{F}$  such that  $\vdash_{\mathcal{E}_{\tilde{f}}^1} \tilde{F} : T_{\tilde{f}}$ , then there is a  $\lambda$ -term  $F$  such that  $\vdash_{\mathcal{E}_{\tilde{f}}^2} F : T_f$ .

**Proof:** This lemma holds both in *AF2* and *TTR*, (using the rules in Table 1). We assume familiarity with *AF2* and only give steps without naming the rules. Let  $K = (D_1 \times \dots \times D_n)$  be the product type of  $f$  with  $cp$  the associated constructor symbol. By definition of the data type  $K$ , we get in *TTR*:

$$a_1 : D_1(x_1), \dots, a_n : D_n(x_n) \vdash_{\mathcal{E}_f} \lambda k(\dots((k a_1) a_2) \dots a_n) : K(cp(x_1, \dots, x_n)).$$

$$\text{Hence: } a_1 : D_1(x_1), \dots, a_n : D_n(x_n) \vdash_{\mathcal{E}_f^1} (\tilde{F} \lambda k(\dots((k a_1) a_2) \dots a_n)) : D(\tilde{f}(cp(x_1, \dots, x_n))).$$

Because  $\mathcal{E}_f^1 \subset \mathcal{E}_f^2$  we have:

$$a_1 : D_1(x_1), \dots, a_n : D_n(x_n) \vdash_{\mathcal{E}_f^2} (\tilde{F} \lambda k(\dots((k a_1) a_2) \dots a_n)) : D(\tilde{f}(cp(x_1, \dots, x_n))).$$

Now, we have the equation  $f(t_1, \dots, t_m) = \tilde{f}(cp(t_1, \dots, t_m))$  in  $\mathcal{E}_f^2$ .

$$\text{Hence: } a_1 : D_1(x_1), \dots, a_n : D_n(x_n) \vdash_{\mathcal{E}_f^2} (\tilde{F} \lambda k(\dots((k a_1) a_2) \dots a_n)) : D(f(x_1, \dots, x_n)).$$

$$\text{Finally: } \vdash_{\mathcal{E}_f^2} F : T_f, \text{ with } F = \lambda a_1 \dots \lambda a_n (\tilde{F} \lambda k(\dots((k a_1) a_2) \dots a_n)). \quad \square$$

We are now going to show that the specification of the twin of a function admits a particular I-proof in our new context according to the fact that its termination is proved with an automated procedure.

## 5.2 Canonical I-proofs

Let  $f : D_1, \dots, D_n \rightarrow D$  be a defined function, with a specification  $\mathcal{E}_f$ , which is terminating with an automated procedure. As mentioned earlier, instead of using the ordering of the terms given in Definition 2.13, we define a new ordering for the symbol relation  $\prec$  by considering the ordering given with the recursive calls of the equations of the specification  $\mathcal{E}_f$ . As in the *ProPre* system, we will assume that we have a subset  $\mathcal{F}_d^*$  of  $\mathcal{F}_d$  of defined functions whose specification admits a proof of totality in *TTR* (the functions already introduced by the user) so that the defined functions occurring in the specification of the function  $f$  for which we want to prove the termination statement, are in  $\mathcal{F}_d^* \cup \{f\}$ .

Now, let  $t$  be a term in  $\mathcal{T}(\mathcal{F}, \mathcal{X})_{s'}$ , for some sort  $s'$  (see Section 3), such that all the defined functions occurring in  $t$  admit a specification and are terminating. Then, for each ground sorted substitution  $\sigma$ , we can define the ground term  $\ulcorner \sigma(t) \urcorner$  as the term in  $\mathcal{T}(\mathcal{F}_c)_s$  that corresponds to the normal form of  $\sigma(t)$ . The definition of  $\ulcorner \sigma(t) \urcorner$  makes sense as the functions occurring in the specification  $f$  are terminating which gives the existence of the normal form while the definition of the specifications (Definition 2.2) gives the uniqueness of the normal form. Therefore, we can state formally the relation  $\prec_{\tilde{f}}$  below.

**Definition 5.5.** Let  $\mathcal{E}_{\tilde{f}}$  be a specification of the twin function of a defined function  $f$  such that the functions occurring in the specification  $\mathcal{E}_{\tilde{f}}$  admit a specification and are terminating. We also assume the function  $f$  to be terminating. Let  $K$  be the product type  $(D_1 \times \dots \times D_n)$  associated to  $f$  and  $cp$  the constructor associated to  $K$ . We define a relation  $\prec_{\tilde{f}}$  on  $K$  such that for each recursive call  $(f(cp(t_1, \dots, t_n)), f(cp(v_1, \dots, v_n)))$  of  $\mathcal{E}_{\tilde{f}}$ , we have  $cp(\ulcorner \sigma(v_1) \urcorner, \dots, \ulcorner \sigma(v_n) \urcorner) \prec_{\tilde{f}} \sigma(cp(t_1, \dots, t_n))$  for any ground sorted substitution  $\sigma$ .

Hence, we get the straightforward but useful following fact.

**Fact 5.6.** The above relation  $\prec_{\tilde{f}}$  is a well-founded ordering on  $K$ .

The next theorem says that if a function  $f$  is terminating and if we have a distributing tree for the specification  $\mathcal{E}_{\tilde{f}}$  of the twin function  $f$ , having or lacking the formal terminal state property, it is then possible to get a new one having the *ftsp*. The principle mainly consists of changing, in the initial distributing tree, the *Struct* and *Ind* rules in such way that we now have a new tree with *ftsp* which can be called a canonical distributing tree. It means that the formal proofs we are going to build will depend on the ability of building

a distributing tree whatever its properties and on the ability of showing the termination of the function.

**Theorem 5.7.** Let  $\mathcal{E}_f$  be a specification of a defined function  $f : D_1, \dots, D_n \rightarrow D$  such that the defined symbols that occur on the right-hand side of the equations of  $\mathcal{E}_f$  are in  $\mathcal{F}_d^* \cup \{f\}$ . Let  $\mathcal{A}$  be a distributing tree for the specification  $\mathcal{E}_{\tilde{f}}$  of the twin function  $\tilde{f}$ . Assume the function  $f$  is proved terminating by a termination procedure. Then there is a distributing tree  $\mathcal{A}'$  for  $\mathcal{E}_{\tilde{f}}$ , which can be automatically obtained from  $\mathcal{A}$ , that satisfies the formal terminal state property with the relation  $\prec_{\tilde{f}}$ .

**Proof:** Let  $\mathcal{E}_f$  be a specification of a defined function  $f : D_1, \dots, D_n \rightarrow s$  such that the defined symbols that occur in the right-hand side of the equations of  $\mathcal{E}_f$  are in  $\mathcal{F}_d^* \cup \{f\}$ . Let  $\mathcal{A}$  be a distributing tree for the specification  $\mathcal{E}_{\tilde{f}}$  of the twin function  $\tilde{f}$ . We assume  $f$  is proved terminating by a termination procedure. Since we know that the function is terminating given by an automated procedure we can introduce the ordering  $\prec_{\tilde{f}}$ . From the term distributing tree  $\mathcal{A}$  we can associate a new distributing tree  $\mathcal{A}'$  with the ordering  $\prec_{\tilde{f}}$ , illustrated with Figure 4, which can be called the canonical distributing tree of  $\mathcal{A}$ .

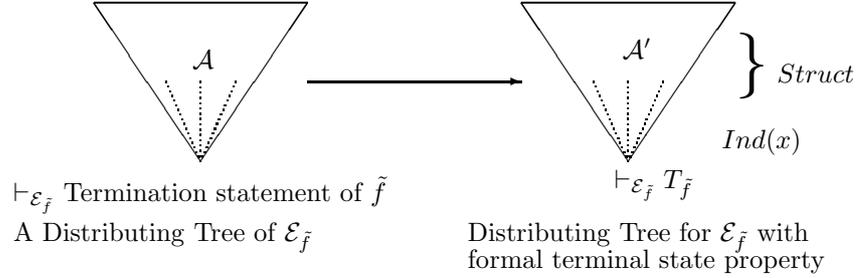


Figure 4: The canonical distributing tree  $\mathcal{A}'$  of  $\mathcal{A}$

Note that  $\mathcal{A}'$  can be built automatically from  $\mathcal{A}$ . We show that  $\mathcal{A}'$  satisfies the formal terminal state property. The root of  $\mathcal{A}'$  is  $\vdash_{\mathcal{E}_{\tilde{f}}} T_{\tilde{f}}$ , with  $T_{\tilde{f}} = \forall x(K(x) \rightarrow D(\tilde{f}(x)))$  the termination statement of  $\tilde{f}$  where  $K$  denotes the product type  $(D_1 \times \dots \times D_n)$  and  $cp$  its associated constructor.

Let  $L = (\Gamma \vdash_{\mathcal{E}_{\tilde{f}}} P)$  be a leaf of  $\mathcal{A}'$  and  $e = (t, u)$  be the equation of  $\mathcal{E}_{\tilde{f}}$  with  $H(P) = t$ . Let  $(t, v)$  be a recursive call of  $e$ . According to the definition of a specification and a recursive call, the terms  $t$  and  $v$  are respectively of the form  $f(cp(t_1, \dots, t_n))$  and  $f(cp(v_1, \dots, v_n))$ . Because of the construction of the canonical distributing tree  $\mathcal{A}'$  that uses a particular order of the application rules *Struct* and *Ind* (also illustrated with Figure 4),  $P$  is of the form:

$$\forall x'_{i_1} D'_{i_1}(x'_{i_1}), \dots, \forall x'_{i_m} D'_{i_m}(x'_{i_m}), \forall z(Kz_{\prec_{cp}(h_1, \dots, h_n)} \rightarrow K(f(z))) \\ \rightarrow K(f(cp(h_1, \dots, h_n))).$$

As the heart of  $P$  is  $H(P) = t$ , we have  $h_j = t_j$  for any  $1 \leq j \leq n$ . Now, let  $Q$  be the restrictive hypothesis  $\forall z(Kz_{\prec_{cp}(t_1, \dots, t_n)} \rightarrow K(f(z)))$  of  $P$ . Let us show that  $Q$  can be applied to the term  $v$  according to a substitution. By the definition of  $Q$ , we have  $H(Q) = f(z)$ , so we can take a substitution  $\sigma$  with  $\sigma(z) = cp(v_1, \dots, v_n)$ . We also take the value  $\sigma(y) = y$  for any free variable  $y$  in  $Q$ , that is any variable  $y$  in  $cp(t_1, \dots, t_n)$ . Hence  $Q$  can be applied to  $v$  according to the above substitution  $\sigma$ . We now have to show the two items of Definition 2.14. As we are in the conditions of Definition 5.5, we know that  $cp(\ulcorner \rho(v_1) \urcorner), \dots, \ulcorner \rho(v_n) \urcorner) \prec_{\tilde{f}} \rho(cp(t_1, \dots, t_n))$  for any ground sorted substitution  $\rho$ . But  $\sigma(z) = cp(v_1, \dots, v_n)$ , thus we get the first item. The second item becomes straightforward:

because of the form of  $Q$ , the set of restrictive hypotheses of  $Q$  is empty. Hence, we conclude that the canonical distributing tree  $\mathcal{A}'$  satisfies the formal terminal state property.  $\square$

The next theorem (and its proof) expresses Figure 3. It tells that if we know that a function  $f$  is terminating, and if we have already a proof of totality of each defined function that occurs in the specification of  $f$  (apart from  $f$ ), and if we have a term distributing tree associated to the specification of  $f$ , then we are able to get a  $\lambda$ -term that computes the function  $f$  in the sense of *TTR*.

**Theorem 5.8.** Let  $\mathcal{E}_f$  be a specification of a defined function  $f : D_1, \dots, D_n \rightarrow D$  and  $\mathcal{D}$  be a given distributing tree for the specification  $\mathcal{E}_f$  such that the defined symbols that occur on the right-hand side of the equations of  $\mathcal{E}_f$  are in  $\mathcal{F}_d^* \cup \{f\}$ . Assume the termination of the function  $f$  given by an automated procedure. Then there is a proof of totality of  $f$  in *TTR* that can be found automatically.

**Proof:** Let  $\tilde{f}$  be the twin function of  $f$  and  $\mathcal{E}_{\tilde{f}}$  its specification given in Definition 5.3. By Definition 5.3, a distributing tree  $\mathcal{A}$  associated to  $\mathcal{E}_{\tilde{f}}$  can be automatically obtained from  $\mathcal{D}$ . Hence, with Theorem 5.7, we now have a (canonical) distributing tree  $\mathcal{A}'$  associated to  $\mathcal{E}_{\tilde{f}}$  which has the *ftsp* with  $\prec_{\tilde{f}}$  as the ordering relation. As Fact 2.11 still holds with the new ordering relation, we get an I-proof of  $\mathcal{E}_{\tilde{f}}$  that can be called *canonical proof*. Thus we obtain a formal proof of the termination statement  $T_{\tilde{f}}$  in *TTR*. Hence, by Lemma 5.4 we finally obtain a proof of totality of  $f$  in *TTR*.  $\square$

Let us now go back to Example 5.1. It was shown in [1, 4] that the termination of the specification  $\mathcal{E}_{quot}$  can be proven with automated termination methods. So if we consider such methods in the right upper part of Figure 3, we then obtain a new ordering relation  $\prec$  for the specification  $\mathcal{E}_{\widetilde{quot}}$  of the associated function  $\widetilde{quot}$ . Together with our setting, this provides a formal proof of totality of  $\widetilde{quot}$  as expressed in the left part of Figure 3. Finally, using this latter result, and thanks to Lemma 5.4, we obtain a formal proof of totality of the function *quot* which was not previously possible in the *ProPre* system .

## 6 Conclusion

An important part of the programming paradigms using logics as is done in *ProPre*, is the Curry-Howard isomorphism where a  $\lambda$ -term is extracted from the proof. However because of the logical framework, it is often difficult to make use of termination techniques from different areas. The study of this paper has shown that, for the automated system *ProPre*, the extraction part of  $\lambda$ -terms can be released from the termination analysis, using the setting of *ProPre*, so that other automated termination techniques (like those of [1, 2, 4]) can now be included in this framework modulo distributing trees.

## References

- [1] T. Arts and J. Giesl. Automatically proving termination where simplification orderings fail. In *Proceedings of Theory and Practice of Software Development TAPSOFT'97*, volume 1214 of *LNCS*, pages 261–272, 1997.
- [2] J. Giesl. Termination of nested and mutually recursive algorithms. *J. of Automated Reasoning*, 19:1–29, 1997.
- [3] W. A. Howard. The formulæ-as types notion of construction. In J. Hindley and J. Seldin, editors, *To H.B. Curry: Essays on combinatory logic, lambda-calculus and formalism*, pages 479–490. Academic Press, 1980.

- [4] F. Kamareddine and F. Monin. On automating inductive and non-inductive termination methods. In *Proceedings of the 5th Asian Computing Science Conference*, volume 1742 of *LNCS*, pages 177–189, 1999.
- [5] F. Kamareddine and F. Monin. On formalised proofs of termination of recursive functions. In *Proceedings of the Int. Conf. on Principles and Practice of Declarative Programming*, volume 1702 of *LNCS*, pages 29–46, 1999.
- [6] F. Kamareddine, F. Monin and M. Ayala-Rincón. On automating the extraction of programs from proofs using product types. In *Proceedings of the 9th Workshop on Logic, Language, Information and Computation, WoLLIC'2002*, Volume 67 of *ENTCS*, 20 pages, 2002.
- [7] J. L. Krivine. *Lambda-calculus, Types and Models*. Computers and Their Applications. Ellis Horwood, 1993.
- [8] J. L. Krivine and M. Parigot. Programming with proofs. *J. Inf. Process Cybern*, 26(3):149–167, 1990.
- [9] D. Leivant. Typing and computational properties of lambda expression. *Theoretical Computer Science*, 44:51–68, 1986.
- [10] P. Manoury. A user’s friendly syntax to define recursive functions as typed lambda-terms. In *Proceedings of Type for Proofs and Programs TYPES'94*, volume 996 of *LNCS*, pages 83–100, 1994.
- [11] P. Manoury, M. Parigot, and M. Simonot. ProPre, a programming language with proofs. In *Proceedings of Logic Programming and Automated Reasoning*, volume 624 of *LNCS*, pages 484–486, 1992.
- [12] P. Manoury and M. Simonot. *Des preuves de totalité de fonctions comme synthèse de programmes*. PhD thesis, University Paris 7, 1992.
- [13] P. Manoury and M. Simonot. Automatizing termination proofs of recursively defined functions. *Theoretical Computer Science*, 135(2):319–343, 1994.
- [14] F. Monin and M. Simonot. An ordinal measure based procedure for termination of functions. *Theoretical Computer Science*, 254(1-2):63–94, 2001.
- [15] M. Parigot. Recursive programming with proofs: a second type theory. In *Proceedings of the European Symposium on Programming ESOP'88*, volume 300 of *LNCS*, pages 145–159, 1988.
- [16] M. Parigot. Recursive programming with proofs. *Theoretical Computer Science*, 94(2):335–356, 1992.